DETERMINATION OF ALL CLASSES OF POSITIVE QUATERNARY QUADRATIC FORMS WHICH REPRESENT ALL (POSITIVE) INTEGERS[1]

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1. **Introduction.** It was apparently known to Diophantus and first proved by Lagrange [8][2] that the form $x^2+y^2+z^2+u^2$ represents all positive integers. Examples of other integral forms

$$(1.1) \qquad \Phi = ax^2 + by^2 + cz^2 + du^2$$

which represent all positive integers were first obtained by Jacobi [4], Liouville [9], and Pepin [13]. Ramanujan [14] proved that there are only 54 sets of positive integers a, b, c, d, such that (1.1) represents all positive integers. Dickson called all such forms universal [3].

Among the 54 universal forms of type (1.1) the form $x^2+2y^2+4z^2+14u^2$ has the largest determinant, 112 [1].

It is of interest to ask what would be the largest determinant of the universal form if we should allow cross product terms. If we should consider forms whose cross product terms may have odd coefficients then the value of the determinant may not exceed 861 [17], and is of that order of magnitude. If, however, we should consider only the so-called "classic" forms, that is, forms whose cross product terms have necessarily even integral coefficients, then we find the value of the largest determinant of such universal forms is, remarkably enough, again 112 [15].

The number of universal forms without cross products, whose coefficients are arranged in the order of increasing magnitude, is equal to 54 [2]. Since such forms are reduced forms we may rephrase this result to say that the number of classes of forms which are equivalent to a form without cross products is equal to 54.

Pursuing this idea further we may ask what is the number of classes of classic universal forms, or extending still farther the set of admissible forms, we may attempt to determine the number of universal classes of non-classic integral forms. In the last case this number is very large as one can see from the work of Morrow [10]. However since in the case of classic forms the upper bound for the determinant is small (112), one should expect that the number of uni-

[2] Numbers in brackets refer to the bibliography at the end of the paper.