SYMMETRY OF ALGEBRAS OVER A NUMBER FIELD

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1. **Introduction.** If the field N is a finite normal extension of the field k, and if K is a normal subfield with $N \supset K \supset k$, a fundamental theorem of Galois theory asserts that every automorphism λ of K over k can be extended to an automorphism of N. As Teichmüller in [7][1] and Jacobson [6, p. 36] have shown, the development of a Galois theory for a simple algebra A with center K leads naturally to a related question: can a given automorphism λ of K be extended to an automorphism of the algebra A? In the event that all automorphisms λ of a finite group Q of automorphisms of K are so extendable, we say that the algebra A is Q-normal. Since any total matric algebra over K is Q-normal for any Q, it follows that any algebra A similar to a Q-normal algebra is Q-normal, and hence that "Q-normality" is a property of algebra classes. Furthermore, if k is the subfield of all elements of K invariant under each automorphism λ of Q, any simple algebra B with center k yields a scalar extension B_K with center K which is Q-normal. The algebra class of any B_K (that is, the algebra classes obtained by scalar extension from k) may thus be termed *trivially* Q-normal. The further investigation of these properties thus raises the problem: are there any algebras which are Q-normal but not trivially so?

If $K \supset k$ are p-adic fields, Köthe [5] has shown that every algebra class over K may be obtained by scalar extension from k, so that in this case all Q-normal algebra classes are trivial. If K is an algebraic number field, he shows that there are algebra classes over K which cannot be obtained by scalar extension. If Q is cyclic, and if K is an algebraic number field, Deuring [2] showed that every Q-normal algebra class is trivially Q-normal. By using three-dimensional cocycles, the same results may be proved for Q cyclic and any field K (Teichmüller, op. cit. p. 149 or Eilenberg-MacLane [3, Corollary 7.3]). In case Q is not cyclic, the answer to our question apparently depends on the arithmetic properties of the field K. In case K is an algebraic number field, the algebra classes can be described completely by the usual arithmetic invariants (cf. for example, Deuring [1, chap. VII]). Using these invariants and the above facts about the cyclic case we obtain in Theorem 3 a complete description of the

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[1] Numbers in brackets refer to the bibliography at the end of the paper.