THE KOLMOGOROFF PRINCIPLE FOR THE LEBESGUE AREA

R. G. HELSEL AND E. J. MICKLE

1. **Introduction.** Kolmogoroff [3][1] has considered an axiomatic development of measure theory for analytic sets in a metric space. One of his axioms is the following: If the set E^* is the image of a set E under a Lipschitzian transformation with Lipschitz constant equal to or less than one, then the measure of E^* should not exceed the measure of E. As stated, the principle has meaning only for point sets, but we shall be concerned with the area (or two-dimensional measure) of Fréchet surfaces in Euclidean 3-space; so we first reformulate the Kolmogoroff principle as follows: If T^* and T are two continuous transformations from the unit disc D: $u^2+v^2\leqq 1$ into Euclidean 3-space which satisfy the distance inequality $|T^*(p_1)-T^*(p_2)|\leqq|T(p_1)-T(p_2)|$ for every pair of points $p_1, p_2\in D$, then the area of the Fréchet surface determined by T^* should not exceed the area of the one determined by T. The purpose of this paper is to show that the Kolmogoroff principle, as reformulated above for Fréchet surfaces, is satisfied by the Lebesgue area.

We first discuss the case in which both the Fréchet surfaces are represented by Lipschitzian transformations. Since the Lebesgue area is given by the usual double integral formula in this case (see Radó [5]), an elementary inequality suffices to establish the desired relation between the areas. In an independent study of a related question, Reichelderfer [6] has obtained a result which can be shown to be equivalent. The general case, in which the transformations representing the surfaces are merely assumed to be continuous, depends upon the extension of a Lipschitzian vector function, defined originally on a bounded and closed set, to an open neighborhood of the set. We are indebted to Professor S. Eilenberg for calling our attention to the fact that such an extension has already been established by Kirszbraun [2].

2. **The Lipschitz case.** To begin, let us establish the elementary inequality mentioned in the introduction.

LEMMA 1. *If E, F, G and E^*, F^*, G^* are constants such that*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.