# NICHOLSON'S INTEGRAL FOR $J_n^2(z)+Y_n^2(z)$

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The integral in question is

$$(1)\qquad J_n^2(z)+Y_n^2(z) = (8/\pi^2)\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt,$$

and its validity for arbitrary complex $n$ when the real part of $z$ is positive is proved in [1, pp. 441–444][1] with the help of Hardy's theory of generalized integrals and integrations over contours in the complex plane. It is the purpose of this paper to give a much more elementary proof of (1).

We begin by observing [1, p. 146] that if $D=z(d/dz)$, then three linearly independent solutions of the equation

$$(2)\qquad [D(D^2-4n^2)+4z^2(D+1)]y=0$$

are $J_n^2(z)$, $Y_n^2(z)$ and $J_n(z)Y_n(z)$. Equation (2) may be written as

$$(3)\qquad z^2y'''+3zy''+(1-4n^2+4z^2)y'+4zy=0.$$

We shall now show that $y(z)=\int_0^\infty K_0(2z\sinh t)\cosh 2nt\,dt$ is a solution of (3). When the real part of $z$ is positive it is clear that $K_0(2z\sinh t)$ is sufficiently small at $\infty$ to permit us to differentiate under the integral sign as many times as we please. Therefore,

$$(4)\qquad y'(z) = 2\cdot\int_0^\infty \sinh t K_0'(2z\sinh t)\cosh 2nt\,dt.$$

If we make use of the differential equation

$$(5)\qquad xK_0''(x)+K_0'(x)-xK_0(x)=0$$

satisfied by $K_0(x)$, then we find that

$$y'' = \int_0^\infty \{4\sinh^2 tK_0(2z\sinh t)-2z^{-1}\sinh tK_0'(2z\sinh t)\}\cosh 2nt\,dt,$$

$$y''' = \int_0^\infty \{(8\sinh^3 t+4z^{-2}\sinh t)K_0'(2z\sinh t) - 4z^{-1}\sinh^2 tK_0(2z\sinh t)\}\cosh 2nt\,dt.$$

It follows that

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[1] Numbers in brackets refer to the reference cited at the end of the paper.