NOTE ON THE LOCATION OF THE CRITICAL POINTS OF HARMONIC FUNCTIONS

J. L. WALSH

By a limiting process, a theorem recently proved by the writer can be generalized, and yields a new result with interesting applications which we wish to record here. We take as point of departure[1] the following theorem.

THEOREM 1. *Let the region R of the extended (x, y)-plane be bounded by a finite number of mutually disjoint Jordan curves $C_0, C_1, C_2, \cdots, C_n$. Let the function $u(x, y)$ be harmonic in R, continuous in the corresponding closed region, equal to zero on C_0 and to unity on $C_1, C_2, \cdots, C_n$. Denote by R_0 the region bounded by C_0 containing the curves $C_1, C_2, \cdots, C_n$ in its interior; define noneuclidean straight lines in R_0 as the images of arcs of circles orthogonal to the unit circle, when R_0 is mapped conformally onto the interior of the unit circle.*

If Π is any non-euclidean convex region in R_0 which contains all the curves $C_1, C_2, \cdots, C_n$, then Π also contains all critical points of $u(x, y)$ in R.

We extend Theorem 1 by admitting arcs of C_0 on which $u(x, y)$ is prescribed to take the value unity, and also by admitting the intersection of curves $C_1, C_2, \cdots, C_n$ with C_0:

THEOREM 2. *Let the region R be bounded by the whole or part of the Jordan curve C_0, and by mutually disjoint Jordan arcs or curves $C_1, C_2, \cdots, C_n$ in the closed interior of C_0; some or all of the latter arcs or curves may have points in common with C_0. Let a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ of C_0 belong to the boundary of R and be mutually disjoint. Let the function $u(x, y)$ be harmonic and bounded in R, and take continuously the boundary values unity on $C_1, C_2, \cdots, C_n, \alpha_1, \alpha_2, \cdots, \alpha_m$ and zero in the remaining boundary points of R, except that in points common to C_0 and $C_1+C_2+ \cdots +C_n$ and in end points of the α_j, no continuous boundary value is required. Denote by R_0 the region bounded by C_0 containing R, and define non-euclidean straight lines in R_0 by mapping R_0 onto the interior of a circle. If Π is any closed region in the closure of R_0 which is non-euclidean convex and which contains $C_1+C_2+ \cdots +C_n+\alpha_1+\alpha_2+ \cdots +\alpha_m$, then Π contains all critical point of $u(x, y)$ in R.*

Presented to the Society, April 26, 1947; received by the editors April 21, 1947.

[1] Bull. Amer. Math. Soc. vol. 52 (1946) pp. 346–347.