ON GREEN'S AND NEUMANN'S FUNCTIONS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

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Introduction. The purpose of this paper is to describe an effective construction of Green's and Neumann's functions for a general class of linear, second-order partial differential equations of elliptic type in terms of a set of continuously differentiable functions, complete and orthonormal with respect to the domain considered. This result completes a previous paper[2] by us in which an analogous construction was given for the difference between the Green and Neumann functions. In order to obtain representations of the Green and Neumann functions separately, one had to assume in the former paper the knowledge of a fundamental solution of the differential equation considered. Since it is by no means an easy task to construct for a given linear partial differential equation of second order a fundamental solution, and since the method of this paper permits a direct construction of the Green and Neumann functions without knowledge of such a fundamental solution, our result seems to constitute a certain advance in the application of orthonormal sets to the theory of linear partial differential equations.

1. **Generalities and notations.** We choose a finite domain B in the plane (x, y) which is bounded by a set of M closed smooth curves C_ν $(\nu=1, 2, \cdots, M)$. We shall denote the boundary of B by $C=\sum_{\nu=1}^{M} C_\nu$. Let $P(x, y)$ be a continuous positive function in the closed region $B+C$. Consider the linear partial differential equation of elliptic type

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=P(x, y)\phi, \qquad P(x, y)>0. \tag{1}$$

We associate with this equation the following Dirichlet integral

$$D\{\phi, \psi\}=\iint_B\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}+P\phi\psi\right)dxdy, \tag{2}$$

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[2] S. Bergman and M. Schiffer, *A representation of Green's and Neumann's functions . . .*, Duke Math. J. vol. 14 (1947).