AN ELECTRO-MECHANICAL INVESTIGATION OF THE RIEMANN ZETA FUNCTION IN THE CRITICAL STRIP

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The theoretical basis underlying the following electromechanical investigations concerns a simple integral representation of the Riemann zeta function in the critical strip. To the right of the critical strip ($\operatorname{Re} s > 1$) we have the usual expression:

$$(1) \qquad \Gamma(s)\cdot\zeta(s) = \int_0^\infty \frac{1}{e^x - 1}\, x^{s-1} dx, \qquad \operatorname{Re} s > 1.$$

Subtraction from (1) of

$$(2) \qquad \Gamma(s-1)\cdot\alpha^{1-s} = \int_0^\infty e^{-\alpha x} x^{s-2} dx, \qquad \operatorname{Re} s > 1,\ \operatorname{Re}\alpha > 0,$$

yields

$$(3)\ \Gamma(s)\cdot\left\{\zeta(s) - \frac{\alpha^{1-s}}{1-s}\right\} = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{e^{-\alpha x}}{x}\right) x^{s-1} dx, \quad \operatorname{Re} s > 0.$$

In the integrand of (3) the simple pole at $x=0$ of $(e^x-1)^{-1}$ is compensated by the simple pole with the same residue of $x^{-1}e^{-\alpha x}$. Therefore (3) already converges in the wider band $\operatorname{Re} s>0$. If we now restrict s to the critical strip

$$0 < \operatorname{Re} s < 1$$

we can let $\alpha\to 0$ in (3), so that we obtain the basic integral

$$(4) \qquad \Gamma(s)\cdot\zeta(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x}\right) x^{s-1} dx, \quad 0 < \operatorname{Re} s < 1,$$

this being the analytical continuation of (1).

The representation (4) of the zeta-function in the critical strip enables us to derive the functional equation of the zeta-function in an extremely simple way. To this end we make use of Legendre's relation[1]

$$(5) \quad 2\int_0^\infty \sin xt \frac{1}{e^{2\pi t} - 1}\, dt = \frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2} = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x},$$

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[1] See, for example, Whittaker-Watson, *Modern analysis*, 4th ed., p. 122.