PRIMITIVE RECURSIVE FUNCTIONS

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1. **Definition of recursive functions.** In this paper, we shall consider certain reductions in the recursion scheme for defining primitive recursive functions. Hereafter, we shall refer to such functions simply as recursive functions.[1] In §1, we define what is meant by a recursive function, and also define some recursive functions which will be used. The statement of the principal results of the paper will be found in §2.

By a number, we shall mean one of the natural numbers 0, 1, 2, 3, $\cdots$. We shall consider functions of any number of variables, each variable ranging over all numbers, and the values of the function being numbers. Small letters will denote variables assuming numerical values, and capital letters will denote functions. In the case of a function of one variable, we shall usually write Fx instead of $F(x)$.

A function will be called *recursive* if it can be obtained from certain initial functions by repeated substitution and recursion.

The initial functions are the following:

The identity functions; that is, for every n and k with $1 \leqq k \leqq n$, the function I_{nk} defined by

$$I_{nk}(x_1, \cdots, x_n) = x_k.$$

The zero functions; that is, for every $n \geqq 0$, the function O_n defined by

$$O_n(x_1, \cdots, x_n) = 0.$$

The successor function; that is, the function S of one variable, such that Sx is the next number after x.

The substitution rule is the following: If $A_1, \cdots, A_m$ are known functions of n variables, and B is a known function of m variables,

Presented to the Society, August 22, 1946; received by the editors March 5, 1947.

[1] Various types of recursive functions play a fundamental role in mathematical logic. Primitive recursive functions were used by K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* I, Monatshefte für Mathematik und Physik vol. 38 (1931) pp. 173–198; for the definition, see pp. 179–180. For a discussion of general recursive functions (which may be considered to be the computable functions), see S. C. Kleene, *General recursive functions of natural numbers*, Math. Ann. vol. 112 (1936) pp. 727–742. For some other types of recursion, see the paper by R. Péter cited in footnote 2. In the present paper, we consider only the simplest type of recursive function, the primitive recursive function, and no knowledge of other papers is assumed.