SOME INEQUALITIES RELATING TO CONFORMAL MAPPING UPON CANONICAL SLIT-DOMAINS

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Let D be a domain of the extended z-plane ($z=x+iy$) of finite connectivity n, which contains the point $z=\infty$ and is bounded by n proper[1] continua. According to a fundamental theorem in the theory of conformal mapping of multiply-connected domains [4, 7][2] there exists one and only one function $\zeta=s_\theta(z)$ which in the neighborhood of $z=\infty$ has a Laurent expansion of the form

$$\zeta = s_\theta(z) = z + \frac{a_\theta}{z} + \cdots \tag{1}$$

and which maps D conformally and bi-uniformly upon a domain D_θ of the ζ-plane bounded by n rectilinear slits each of which makes the angle θ with the positive direction of the real axis. The domain D_θ is itself also uniquely determined for each value of θ.

In the present paper we shall derive two inequalities involving the coefficient a_θ appearing in (1) and the outer measure A of the complement (with respect to the entire plane) of the domain D—that is, the greatest lower bound of the total area enclosed by a set of analytic curves surrounding the boundary continua. The first of these inequalities is the following:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) \geqq \frac{A}{2\pi}\cdot \tag{2}$$

The second inequality, which will be derived by using the theory of orthonormal systems of analytic functions [1, 2, 9, 10], constitutes a strengthening of (2), namely:

$$\operatorname{Re}\,(a_\theta e^{-2i\theta}) - \frac{|a_\theta|^2}{a_0 - a_{\pi/2}} \geqq \frac{A}{2\pi}\cdot \tag{3}$$

It suffices to prove the inequalities (2) and (3) for the case when the boundary continua of D are closed analytic curves $C_1, C_2, \cdots, C_n$, for it is known that D can be approximated by an increasing sequence of domains having such boundaries for which the mapping functions corresponding to (1) will converge to $s_\theta(z)$, so that (2) and

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[1] A proper continuum is one which does not consist of a single point.

[2] Numbers in brackets refer to the bibliography at the end of the paper.