ON THE CONVERGENCE OF DOUBLE SERIES

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1. **Introduction.** We consider three convergence definitions for double series, denoted by (p), (σ), and (reg), which are respectively Pringsheim, Sheffer, and regular convergence. Definitions will be given in §2.

Convergence (σ) has been defined by I. M. Sheffer in a paper[1] which will be referred to as [S]. Convergence (p) and (reg) are well known, the latter having been discussed by G. H. Hardy[2] and others.

[S] established the relation $(\sigma)\subseteq(\text{reg})$; that is, every series which is convergent (σ) is also convergent (reg) and to the same sum. The question arises as to whether the relation between these two types of convergence is actually equivalence. It is part of the purpose of this paper to answer this question in the negative. An infinite set of convergence definitions will be presented, denoted by (σ_n), $n=1, 2, \cdots$, with the property:

$$(\sigma) \subseteq (\sigma_{n+1}) \subseteq (\sigma_n) \subseteq (\sigma_1) \subseteq (\text{reg}), \qquad n = 1, 2, \cdots,$$

and it will then be shown that every inclusion sign but the first may be replaced by equivalence. Of these the most difficult to prove is $(\sigma_2)\equiv(\sigma_1)$. The others just escape being trivial.

The words "to the same sum" will always be understood in the symbols $\subset$ and $\equiv$.

2. **Definitions.** Definitions 3 to 3.2 are adapted from [S].

DEFINITION 1. *A region is a* finite *set of values of the indices.* Examples of regions are the following:

(i) Triangular region: the set of indices (p, q) with $p+q\leqq N$ for a given N.

(ii) Rectangular region: the set of (p, q) with $p\leqq M$, $q\leqq N$, for given M, N.

By varying N in (i) (M, N, in (ii)) we obtain a set of triangular (rectangular) regions. These two examples have in common the following important property:

PROPERTY (1). *Given any square region containing* (0, 0) *there is a region of the set under consideration which includes the square region.*

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[1] Amer. Math. Monthly vol. 52 (1945) pp. 365–376.

[2] Proc. Cambridge Philos. Soc. vol. 19 (1916–1919) pp. 86–95; Lemmas γ and δ are wrong but Theorem 10 is correct.