NOTE ON POWER SERIES

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1. **The problem.** The following question was raised by Bochner. Let $\sum \alpha_{ik}\xi^i\eta^k$ be a power series with complex coefficients, such that substitution of convergent power series $\sum_1^\infty \alpha_i\zeta^i$ and $\sum_1^\infty \beta_i\zeta^i$ for ξ and η produces always a convergent power series in ζ. Is the double series $\sum \alpha_{ik}\xi^i\eta^k$ convergent?

The answer is yes; we present a proof which presupposes from function theory only the Cauchy estimate for the coefficients of polynomials in a complex variable:

$$\text{(C)} \qquad |\gamma_k\zeta_0^k| \leqq (|\zeta| = |\zeta_0|) \sup |\sum \gamma_k\zeta^k|.$$

We note that this estimate is also valid in certain types of fields with non-Archimedean valuations, namely, those for which the values are dense and the index is infinite; this was shown by Schoebe in [1].[1]

2. **Homogeneous polynomials.** We denote a vector (ξ, η) by x and introduce as the norm $\|x\|$ of x the maximum of $|\xi|$ and $|\eta|$. A complex Banach space results which, as a complete metric space, is of the second category with respect to itself. We then consider homogeneous polynomials $P(x) = \sum_{i+k=n}\alpha_{ik}\xi^i\eta^k$; it is clear that $P(\zeta x) = \zeta^n P(x)$, that $P(x+\zeta x_0)$ is a polynomial in ζ, and that P is a continuous function of x.

The following three lemmata are immediate consequences of the estimate (C).

(2.1) LEMMA. *If* $|P(x)| \leqq M$ *for* $\|x\| \leqq \zeta$, *then* $|\alpha_{ik}\xi^i\eta^k| \leqq M$ *for* $|\xi|, |\eta| \leqq \zeta$.

(2.2) LEMMA. $|P(x)| \leqq (|\zeta| = 1) \sup |P(x+\zeta x_0)|$.

This special case of the principle of the maximum is a special case of (C), applied to the constant term of $P(x+\zeta x_0)$, considered as a polynomial in ζ. It is used in the proof of (2.3).

(2.3) LEMMA. *If* $|P(x)| \leqq M$ *for* $\|x - x_0\| \leqq \zeta$, *then* $|P(x)| \leqq M$ *for* $\|x\| \leqq \zeta$.

PROOF (compare [2, p. 590]): $|P(x)| \leqq (|\zeta| = 1) \sup |P(\zeta x_0 + x)| = (|\zeta| = 1) \sup |P(x_0 + \zeta^{-1}x)| \leqq (\|x_1 - x_0\| \leqq \|x\|) \sup |P(x_1)|$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.