A NOTE ON TRANSFORMS OF UNBOUNDED SEQUENCES

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During an evening session at the recent Ithaca meeting it was conjectured that it is possible to construct a regular Toeplitz matrix $A \equiv \|a_{nk}\|$ with the property that for every sequence s_n the transformed sequence

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k \tag{1}$$

possesses at least one limit point in the finite plane; and it was counter-conjectured that for every regular Toeplitz matrix A there exists a sequence s_n such that the sequence t_n of equation (1) tends to infinity monotonically. It is the purpose of the present note to report that both conjectures are false and to prove a consolation theorem regarding the first conjecture. The notation of equation (1) will be used throughout the paper.

THEOREM 1. *If A is a row-finite regular Toeplitz matrix, there exist sequences s_n such that the corresponding sequences $|t_n|$ tend to infinity with arbitrary rapidity.*

Let A be a row-finite regular Toeplitz matrix. If n_0 is sufficiently large, each row whose index exceeds n_0 contains a nonzero element, and therefore a last nonzero element (a *row-terminal* element). Let $k_1, k_2, \cdots$ be the indices of the columns that contain row-terminal elements ($k_1 < k_2 < \cdots$), and let the terms s_k ($k \neq k_1, k_2, \cdots$) be chosen arbitrarily. Regularity of the matrix A implies that each column contains at most a finite number of row-terminal elements, that is, that for each column the row-terminal elements are bounded away from zero. It is now clear that if $f(n)$ is any arbitrary real function, the terms $s_{k_1}, s_{k_2}, \cdots$ can be chosen large enough so that $|t_n| > f(n)$ ($n > n_0$), and Theorem 1 is proved.

THEOREM 2. *If A is a regular Toeplitz matrix, there exists a sequence s_n such that the sequence t_n has no limit point in the finite plane.*

Here the matrix A is not required to be row-finite, and if the sequence s_n is chosen (as in the proof of Theorem 1) so as to tend to infinity with reckless rapidity, there is danger that the transformation A does not apply to the sequence (*the transformation A applies to the*

Received by the editors October 7, 1946, and, in revised form, March 7, 1947.