MERTENS' THEOREM AND SEQUENCE TRANSFORMATIONS

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The purposes of this note are to prove by sequence transformation theory a known [1, Theorem 6][1] form of Mertens' theorem which admits of a valid converse, and by extending this method to supplement some recent results of Sheffer [2] on extensions of Mertens' theorem to higher dimensions.

Let $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ be two convergent series, with sums A and B, respectively. We shall refer to these series as "a-series" and "b-series," respectively. By definition, their "Cauchy-product series" is $\sum_{k=0}^{\infty}\sum_{i=0}^{k} a_i b_{k-i}$. We shall refer to this series as the "c-series" for the a- and b-series. Now Mertens' theorem states that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series converge absolutely.* Examples show that the condition is not necessary. On the other hand, J. D. Hill has rephrased the theorem so that its converse *is* true.[2]

THEOREM 1 *A necessary and sufficient condition that the c-series for an a-series and each b-series converge to AB is that the a-series converge absolutely.*

In two dimensions our a-, b-, and c-series become, respectively, $\sum_{i,j=0}^{\infty} a_{ij}$, $\sum_{i,j=0}^{\infty} b_{ij}$, and $\sum_{k,l=0}^{\infty}\sum_{i,j=0}^{k,k} a_{ij} b_{k-i,l-j}$. We shall again suppose that the first two converge,[3] and shall denote their sums by A and B, respectively. Sheffer has recently shown [2, Theorem 1] that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series converge absolutely and the other series converge boundedly.*[4] In our Theorem 3 (below) we show that *a sufficient condition that the c-series converge to AB is that either the a- or the b-series have only a finite number of nonzero terms.* Sheffer's results and ours may be regarded as generalizations of Mertens' theorem, his being the more interesting and ours at first sight trivial.

More interesting than either generalization is the problem of rephrasing it so as to obtain a valid converse. Sheffer in fact does this

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[1] Numbers in brackets refer to the papers cited at the end of this article.

[2] We defer proofs to the end.

[3] Convergence of multiple series shall be in the sense of Pringsheim.

[4] That is, with bounded partial sums.