A PRIME-REPRESENTING FUNCTION

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A function $f(x)$ is said to be a prime-representing function if $f(x)$ is a prime number for all positive integral values of x. It will be shown that there exists a real number A such that $[A^{3^x}]$ is a prime-representing function, where $[R]$ denotes the greatest integer less than or equal to R.

Let p_n denote the nth prime number. A. E. Ingham[1] has shown that

$$p_{n+1} - p_n < Kp_n^{5/8} \tag{1}$$

where K is a fixed positive integer.

LEMMA. *If N is an integer greater than K^8 there exists a prime p such that $N^3 < p < (N+1)^3 - 1$.*

PROOF. Let p_n be the greatest prime less than N^3. Then

$$\begin{aligned} N^3 < p_{n+1} < p_n + Kp_n^{5/8} < N^3 + KN^{15/8} < N^3 + N^2 \\ < (N+1)^3 - 1. \end{aligned} \tag{2}$$

Let P_0 be a prime greater than K^8. Then by the lemma we can construct an infinite sequence of primes, $P_0, P_1, P_2, \cdots$, such that $P_n^3 < P_{n+1} < (P_n+1)^3 - 1$. Let

$$u_n = P_n^{3^{-n}}, \qquad v_n = (P_n + 1)^{3^{-n}}. \tag{3}$$

Then

$$v_n > u_n, \qquad u_{n+1} = P_{n+1}^{3^{-n-1}} > P_n^{3^{-n}} = u_n, \tag{4}$$

$$v_{n+1} = (P_{n+1} + 1)^{3^{-n-1}} < (P_n + 1)^{3^{-n}} = v_n. \tag{5}$$

It follows at once that the u_n form a bounded monotone increasing sequence. Let $A = \lim_{n\to\infty} u_n$.

THEOREM. *$[A^{3^x}]$ is a prime-representing function.*

PROOF. From (4) and (5) it follows that $u_n < A < v_n$, or $P_n < A^{3^n} < P_n + 1$.

Therefore $[A^{3^n}] = P_n$ and $[A^{3^x}]$ is a prime-representing function.

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[1] A. E. Ingham, *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 255–266.