INNER PRODUCTS IN NORMED LINEAR SPACES

ROBERT C. JAMES

Let T be any normed linear space [1, p. 53].[1] Then an *inner product* is defined in T if to each pair of elements x and y there is associated a real number (x, y) in such a way that $(x, y) = (y, x)$, $\|x\|^2 = (x, x)$, $(x, y+z) = (x, y) + (x, z)$, and $(tx, y) = t(x, y)$ for all real numbers t and elements x and y. An inner product can be defined in T if and only if any two-dimensional subspace is equivalent to Cartesian space [5]. A complete separable normed linear space which has an inner product and is not finite-dimensional is equivalent to (real) Hilbert space,[2] while every finite-dimensional subspace is equivalent to Euclidean space of that dimension. Any complete normed linear space T which has an inner product is characterized by its (finite or transfinite) cardinal "dimension-number" n. It is equivalent to the space of all sets $x = (x_1, x_2, \cdots)$ of n real numbers satisfying $\sum_i x_i^2 < +\infty$, where $\|x\| = (\sum_i x_i^2)^{1/2}$ [7, Theorem 32]. Various necessary and sufficient conditions for the existence of an inner product in normed linear spaces of two or more dimensions are known. Two such conditions are that $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$ for all x and y, and that $\lim_{n\to\infty} \|x+ny\| - \|nx+y\| = 0$ whenever $\|x\| = \|y\|$ ([5] and [4, Theorem 6.3]). A characterization of inner product spaces of three or more dimensions is that there exist a projection of unit norm on each two-dimensional subspace [6, Theorem 3]. Other characterizations valid for three or more dimensions will be given here, expressed by means of orthogonality, hyperplanes, and linear functionals.

A hyperplane of a normed linear space is any closed maximal linear subset M, or any translation $x+M$ of M. A hyperplane is a supporting hyperplane of a convex body S if its distance from S is zero and it does not contain an interior point of S; it is tangent to S at x if it is the only supporting hyperplane of S containing x [8, pp. 70–74]. It will be said that an element x_0 of T is orthogonal to y ($x_0 \perp y$) if and only if $\|x_0 + ky\| \geqq \|x_0\|$ for all k, which is equivalent to requiring the existence of a nonzero linear functional f such that $f(x_0) = \|f\| \, \|x_0\|$ and $f(y) = 0$, or that $x_0 + y$ belong to a supporting hyperplane of the sphere

Presented to the Society, November 2, 1946; received by the editors December 31, 1946.

[1] Numbers in brackets refer to the references at the end of the paper.

[2] "Equivalent" meaning isometric under a linear transformation [1, p. 180]. The equivalence to (real) Hilbert space follows by reasoning similar to that of [10, pp. 3–16].