ON REAL CONTINUOUS SOLUTIONS OF ALGEBRAIC DIFFERENCE EQUATIONS

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1. Introduction. This paper considers the behaviour at infinity of real continuous solutions of algebraic difference equations

$$P(y(x+m), \cdots, y(x), x) = 0 \tag{1}$$

where P is a polynomial with real coefficients in its arguments $y(x+m), \cdots, y(x)$ and x. The problem was first treated by Lancaster,[1] who obtained an upper bound for the rate of increase of the solutions of algebraic difference equations of a given order and pointed out the surprising dissimilarity with the known rates of increase for solutions of differential equations of the same order.

The main object of this paper is to show that any real continuous solution of an algebraic difference equation of the first order

$$P(y(x+1), y(x), x) = 0 \tag{2}$$

satisfies the inequality

$$\liminf_{x\to\infty} \log \log |y(x)| / x < \infty. \tag{3}$$

This is an improvement over the results of Lancaster, who proved that a continuous solution $y(x)$ cannot equal or exceed $ce_2(xl_nx)$ for all $x > x_0$, where $l_n(x)$ is the nth iterate of $\log x$, $e_2(x) = e^{e^x}$, n is any fixed positive integer, and c is any positive constant. The essential difference in the rates of increase of the solutions of algebraic differential and difference equations is clearly emphasized when this new result is compared with the results of Borel, who showed that any real continuous solution $y(x)$ of an algebraic differential equation of the first order satisfies the inequality

$$\limsup_{x\to\infty} \log \log |y(x)| / \log x < \infty. \tag{4}$$

Simple examples reveal that relation (3) is the *best possible* result. First this is the best possible limit, for when $y(x) = a^{b^x}$, a solution of

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[1] Otis E. Lancaster, *Some results concerning the behavior at infinity of real continuous solutions of algebraic difference equations*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 169–177. References to the papers of Borel, Vijayaraghavan and others will be found in this paper.