EXTENSIONS OF DIFFERENTIAL FIELDS. III

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The purpose of the present note is to show how the point of view of a preceding paper[1] can be used in developing the concepts of resolvent, dimension, and order introduced by J. F. Ritt in his theory of algebraic differential equations.[2] The present development, in addition to being simpler in some instances, has the advantage of being valid for abstract differential fields as opposed to fields of meromorphic functions of a complex variable, as used by Ritt. I shall also take the opportunity to correct mistakes in a related paper.[3] The notation and definitions used will be as in Extensions I and II.

1. **Resolvents, dimension, and order.** Let $\mathcal{F}$ be a differential field (ordinary or partial) of characteristic 0, and let $y_1, \cdots, y_n$ be unknowns. If Π is a prime differential ideal in $\mathcal{F}\{y_1, \cdots, y_n\}$ other than $\mathcal{F}\{y_1, \cdots, y_n\}$ itself then Π has a generic solution $\eta_1, \cdots, \eta_n$.

If the degree of differential transcendency of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}$ is q then $0 \leqq q < n$, and precisely q of the elements $\eta_1, \cdots, \eta_n$ are differentially algebraically independent over $\mathcal{F}$. Suppose, say, that $\eta_1 \cdots, \eta_q$ are independent in this way, that is, that Π does not contain a nonzero differential polynomial in $y_1, \cdots, y_q$, but does in $y_1, \cdots, y_q, y_j$ for each $j > q$. In Ritt's terminology $y_1 \cdots, y_q$ is a complete set of arbitrary unknowns for Π. It is natural to call q the *dimension* of Π (in symbols, dim Π).

Suppose henceforth that $\mathcal{F}$ is ordinary. It is easy to see that the degree of transcendency of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ (both these differential fields being considered as fields) is finite. We denote the degree of transcendency of any field $\mathcal{K}$ over a subfield $\mathcal{G}$ by $\partial^0\mathcal{K}/\mathcal{G}$. It will be seen that it is natural to call the integer $\partial^0\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle/\mathcal{F}\langle\eta_1, \cdots, \eta_q\rangle$ the *order* of Π with respect to $y_1, \cdots, y_q$ (when the set of arbitrary unknowns is understood, for example when $q=0$, we use the notation: ord Π).

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[1] Kolchin, *Extensions of differential fields*, I, Ann. of Math. vol. 43 (1942) pp. 724–729. We shall refer to this paper as *Extensions* I.

[2] The subject matter treated here, together with some of the material from *Extensions* I, is roughly parallel to the contents of §§24–31, 75 of Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, New York, 1932.

[3] Kolchin, *Extensions of differential fields*, II, Ann. of Math. vol. 45 (1944) pp. 358–361. We shall refer to this paper as *Extensions* II.