THE NON-EXISTENCE OF A CERTAIN TYPE OF ODD PERFECT NUMBER

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For any perfect number[1] expressed in the form $n = a_0 a_1 \cdots a_t$, where

$$a_0 = p_0^{\alpha_0},\ a_1 = p_1^{\alpha_1},\ \cdots,\ a_t = p_t^{\alpha_t}$$

and $p_0, p_1, \cdots, p_t$ are the distinct prime factors of n, it can be shown that a unique one of the prime powers a_i has an even divisor sum $\sigma(a_i)$. Throughout we shall suppose that the primes p_i and hence the prime powers a_i to be so numbered that

$$(1) \qquad \sigma(a_0) \equiv 0; \quad \sigma(a_i) \equiv 1 \qquad i = 1, 2, \cdots, t, \pmod 2.$$

Then with the abbreviations

$$(2) \qquad \sigma_0 = \sigma(a_0)/2; \quad \sigma_i = \sigma(a_i), \qquad i = 1, 2, \cdots, t,$$

the condition for n to be perfect may be written in the form

$$(3) \qquad \sigma(n)/2 = \sigma_0\sigma_1 \cdots \sigma_t = a_0 a_1 \cdots a_t = n.$$

For the even perfect numbers, which are the only kind known, it is well known that $p_0 = 2^q - 1$, $\alpha_0 = 1$, $p_1 = 2$, $\alpha_1 = q - 1$, $t = 1$, where q is any prime such that $2^q - 1$ is also prime. Then $\sigma_1 = 2^q - 1 = a_0$ and $\sigma_0 = 2^{q-1} = a_1$ so that σ_0 and σ_1 are the prime powers a_0 and a_1 in re-reverse order. It is natural to inquire whether there may exist odd perfect numbers such that analogously $\sigma_0, \sigma_1, \cdots, \sigma_t$ are the prime powers $a_0, a_1, \cdots, a_t$ in a different order. In the following it will be proved that no odd perfect numbers of this form can exist.

We first establish an algebraic identity. Throughout this paper the product notation $\prod_{i=a}^{b} x_i$ is used with the convention that $\prod_{i=a}^{b} x_i = 1$ if $a > b$.

LEMMA 1. *Let $c_1, c_2, \cdots, c_t$ be any $t \geqq 2$ integers (more generally, elements of a commutative ring with a unit element). Then,*

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[1] For a summary of results concerning perfect numbers (including those cited above) with references see L. Dickson, *History of the theory of numbers*, vol. 1, 1919, pp. 1–33. For a more recent paper with references to other recent literature on the subject, see A. Brauer, *On the non-existence of odd perfect numbers of form $p^\alpha q_1^2 q_2^2 \cdots q_{t-1}^2 q_t^4$*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 712–718.