ON THE STRUCTURE OF ALGEBRAS WITH NONZERO RADICAL

G. HOCHSCHILD

Introduction. If one attempts to make a systematic study of algebras with nonzero radical one soon realizes that the main difficulty is that of singling out a set of structural characteristics which could constitute a suitable center of interest in a general theory. In the study of simple and semisimple algebras, the full matrix algebras over the groundfield serve as models for the "perfect" structure, and the greatest departure from this which one has to consider consists in the ground field being replaced by a division algebra. On the other hand, even among the nilpotent algebras, it would be difficult to decide what type is to be regarded as representing the "perfect" structure.

The first type that one might think of in this connection is the following: There is a single element, x, with $x^{n+1}=0$, for some positive integer n, whose powers $x, x^2, \cdots, x^n$ constitute a linear basis for the algebra. However, this is evidently far too special a type to serve as a standard. A natural generalization of this, which turns out to be more suitable, is obtained by replacing the one-dimensional subspace, (x), by a subspace of arbitrary dimension. With suitable additional requirements in the case of non-nilpotent algebras, we are led to the notion of a "quasicyclic" algebra whose radical has a considerably more transparent structure than a general nilpotent algebra.

We also introduce a more restrictive notion, that of a "maximal" algebra, and it will be shown that every algebra whose quotient by the radical is separable or $\{0\}$ is the homomorphic image of a "related" maximal (quasicyclic) algebra. Since the structure of maximal algebras is determined completely by separable algebras and their representations our structure theory decomposes into two parts: The study of two-sided ideals contained in the radical of a maximal algebra, and the study of separable algebras and their representations.

Throughout this paper we shall deal only with algebras whose quotients by the radical are either $\{0\}$ or separable. In particular, this will be the case for all algebras over a perfect field. It is to be noted that in this approach there is no need for a general representation theory of non-semisimple algebras.

1. **Related algebras.** We consider algebras B, with radical R, such that B/R is separable or $\{0\}$. Two such algebras, B_1 and B_2, are said

Received by the editors October 24, 1946.