# A NEW PROOF OF HILBERT'S NULLSTELLENSATZ

OSCAR ZARISKI

**Introduction.** A number of proofs of Hilbert's Nullstellensatz can be found in the literature. One, based on elimination theory and due to A. Rabinowitsch, is reproduced in van der Waerden's *Moderne Algebra*, vol. 2, p. 11. In a later chapter van der Waerden gives another proof which is based on the method of specialization in fields of algebraic functions (pp. 59–61). The finishing touches to this proof (p. 65) presuppose the decomposition theorem for polynomial ideals. In his Ergebnisse monograph *Idealtheorie*, p. 46, Krull gives an ideal-theoretic proof which, while it is based on the simple remark by Rabinowitsch, is of an advanced nature, since the proof makes use of the full dimension theory of algebraic varieties developed in §17, pp. 41–43. The main "Dimensionssatz" of p. 43 is based on a result which is proved only in §48, pp. 129–134. Moreover, the concept of integral dependence and the "Normalization theorem" of Emmy Noether are used in Krull's proof.

In the present note we give first of all a short proof of Hilbert's Nullstellensatz which makes use only of the rudiments of field theory and ideal theory. Actually we give two new proofs of the Nullstellensatz. A lemma used in the second proof enables us to establish a result on finite integral domains which we were not able to find in the literature. This result is as follows:

*If $R = \mathrm{K}[\xi_1, \xi_2, \cdots, \xi_n]$ is a finite integral domain over a field* K *and if $\overline{\mathrm{K}}$ is the algebraic closure of* K *in $R$, then $\overline{\mathrm{K}}$ contains all the fields which are contained in $R$.*

1. **First proof of the Nullstellensatz.** Let $\mathfrak{P}_n$ denote the polynomial ring $\mathrm{K}[x_1, x_2, \cdots, x_n]$ in $n$ indeterminates $x_i$, over a given ground field K. By a point $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ we mean an ordered $n$-tuple of algebraic quantities $\alpha_i$ over K. Our convention will be that conjugate $n$-tuples over K represent the same point over K. Let $S_n$ denote the (linear) space of all points. By a zero $\alpha$ of an ideal $\mathfrak{A}$ in $\mathfrak{P}_n$ we mean a point $\alpha$ such that $f(\alpha) = 0$ for every polynomial $f(x)$ $[=f(x_1, x_2, \cdots, x_n)]$ in $\mathfrak{A}$. The totality of zeros of $\mathfrak{A}$ is the algebraic variety in $S_n$ determined by the ideal $\mathfrak{A}$ and shall be denoted by $\mathcal{V}(\mathfrak{A})$.

If $W$ is an algebraic variety in $S_n$, we shall denote by $\mathfrak{J}(W)$ the ideal in $\mathfrak{P}_n$ consisting of all polynomials which vanish on $W$ (that is, at every point of $W$). The Hilbert Nullstellensatz asserts the following:

Received by the editors November 4, 1946.