A NOTE ON THE MEAN VALUE OF THE POISSON KERNEL

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In some investigations it is necessary to evaluate the mean value of some power of the Poisson kernel,

$$P(r, \theta) \equiv (1 - r^2)/(1 - 2r \cos \theta + r^2),$$

with respect to θ. This note gives a closed expression for this mean value, and an exact statement of the order of growth as r approaches 1.

THEOREM 1. *If* $x = 2r/(1+r^2)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r, \theta)d\theta = \left(\frac{1 - r^2}{1 + r^2}\right)^{n+1} \cdot \frac{1}{\Gamma(n + 1)} \cdot \frac{d^n}{dx^n}\left(\frac{x^n}{(1 - x^2)^{1/2}}\right), \qquad n > -1. \tag{1}$$

If n is not an integer the derivative is to be computed by the formula of Riemann and Liouville[1]

$$\frac{d^n}{dx^n}(f(x)) = \frac{d^m}{dx^m} \frac{1}{\Gamma(\rho)}\int_0^x (x - t)^{\rho-1}f(t)dt, \tag{2}$$

where m is the smallest integer not less than n and $\rho = m - n$.

The proof consists merely of the comparison of two power series. Clearly

$$P^{n+1}(r, \theta) = \left(\frac{1 - r^2}{1 + r^2}\right)^{n+1} \left(1 - \frac{2r}{1 + r^2} \cos \theta\right)^{-(n+1)},$$

and the second parenthesis, with $x = 2r/(1+r^2)$, is $1+(n+1)x \cos \theta + (n+1)(n+2)/2!x^2 \cos^2 \theta + \cdots$ by the binomial theorem. Since

$$\int_0^{2\pi} \cos^p \theta d\theta = 0 \qquad \text{(if } p \text{ is an odd integer)}$$

$$= \frac{4(p - 1)(p - 3) \cdots 3\cdot 1}{p(p - 2) \cdots 4\cdot 2}\cdot\frac{\pi}{2} \quad \text{(if } p \text{ is even)}$$

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[1] See, for example, Courant, *Differential and integral calculus*, rev. ed., vol. 2, pp. 339–340.