ON NÖRLUND SUMMABILITY OF RANDOM VARIABLES TO ZERO

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1. **Introduction.** In a previous paper[1] [1], the author considered the Cesàro summability methods $\{C_r\}$ $(0<r<\infty)$ for sequences of independent, real-valued random variables $\{\mathbf{x}_k\}$. For summability in probability of $\{\mathbf{x}_k\}$ to 0, it was shown that: (i) $r<s$ implies $\mathbf{C}_r\subset\mathbf{C}_s$; (ii) for $r\geqq 1$ all the methods $\mathbf{C}_r$ are essentially equivalent, in contrast to the Cesàro theory for sequences of real numbers. The field of the investigation reported here is the summability in probability of sequences $\{\mathbf{x}_k\}$ to 0 by the Nörlund summability methods, which include the Cesàro methods. The objective (attained only in special cases) was to prove that if two Nörlund methods N_p and N_q share the relation $N_p\subset N_q$ over sequences of real numbers, then the analogous relation $\mathbf{N}_p\subset\mathbf{N}_q$ holds for the summability of sequences of independent, real-valued, symmetric random variables to zero. The converse is, of course, false.

The only sequences $\{\mathbf{x}_k\}$ considered here are normal families of independent, real-valued, symmetric random variables. For these $\{\mathbf{x}_k\}$ the objective has been attained for three special cases; see Theorems 4, 5, and 6. The earlier theorems are tools: Theorem 1 gives a necessary and sufficient condition for the Nörlund summability of $\{\mathbf{x}_k\}$ to 0, while Theorems 2 and 3 give sufficient conditions for the relations $\mathbf{N}_p\subset\mathbf{N}_q$ and $\mathbf{N}_p\equiv\mathbf{N}_q$, respectively. Theorem 7 shows that equivalence with $\mathbf{C}_1$ over $\{\mathbf{x}_k\}$ extends to a Nörlund method $\mathbf{N}_p$ whose counterpart N_p is strictly weaker than C_1 over sequences of real numbers. Such equivalence with $\mathbf{C}_1$ is impossible for Cesàro methods weaker than C_1 over sequences of real numbers.

It is conjectured that Theorems 4, 5, and 6, here proved for normal families only, can be extended without change of statement to arbitrary sequences of independent, real-valued, symmetric random variables. If the $\mathbf{x}_k$ are not symmetric there are complications (see [1]), but it is conjectured that Theorems 4, 5, and 6 still hold without essential change.

2. **Nörlund summability of sequences of real numbers.** Let $p=\{p_n\}$ $(n=0, 1, 2, \cdots)$ be a sequence of nonnegative real numbers, with

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[1] Numbers in brackets refer to the references cited at the end of the paper.