ON COMPACT FIBERINGS OF THE PLANE

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In a recent paper [2],[1] Montgomery and Samelson have raised the question whether there exists, for some n, a compact fibering of Euclidean n-space, and have given some reasons for thinking that no such fibering is possible. The purpose of this note is to provide further evidence for this belief by proving that at least there is no compact fibering of the plane. The theorem I shall prove is in fact somewhat stronger.

THEOREM 1. *If $f(E^2)=A$ is an interior*[2] *transformation of the plane such that for each two points, x and y, of A, $f^{-1}(x)$ and $f^{-1}(y)$ are homeomorphic, and such that each component of $f^{-1}(x)$ is compact, then no component of $f^{-1}(x)$ separates E^2 and A is a 2-manifold.*

If in addition $f^{-1}(x)$ is compact, then f is monotone and A is a plane.

PROOF. It cannot have escaped notice that a transformation satisfying the conditions placed on f (omitting the homeomorphy condition) can be factored into a monotone closed transformation, $g(E^2)=B$, followed by a light interior transformation, $h(B)=A$, even though such theorems have been stated for compact spaces only. The arguments for Theorems[3] VIII, 4.1, and VIII, 3.1, establish this. The proof of Theorem IX, 2.3, shows that B is homeomorphic to a set obtained by removing some non-cut point from a cactoid. If some component of some $f^{-1}(x)$ separates E^2, then B has a cut point. Since each set $g^{-1}(x)$ is a component of some set $f^{-1}(y)$, y in A, and each two inverses of points of A are homeomorphic, B has uncountably many cut points. Hence some point p of B is of Menger order 2, by Theorem VII, 3.2. There is a point q of B such that the closure, C, of the component of $B-q$ that contains p is compact. Over C, h is continuous, and over $C-q$, h is interior; indeed, $h(C-q)$ is open in A.

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[1] Numbers in brackets refer to the bibliography.

[2] A continuous transformation $f(X)=Y$ is interior provided that the image of every open subset of X is open in Y, and is light provided that every set $f^{-1}(y)$, y in Y, is totally disconnected.

The statement of this theorem is weaker than that in Bull. Amer. Math. Soc. Abstract 53-1-106. In the original formulation, I used a characterization of the possible interior images of a 2-manifold that I had announced in Bull. Amer. Math. Soc. Abstract 52-5-220, but in the proof of which an error has been found.

[3] Theorems referred to in this way are from Whyburn's book [4]. The roman numeral is the chapter number.