REMARKS ON METRIZABILITY

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In connection with those paragraphs of my paper *Applications of the theory of Boolean rings to general topology* (Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481) dealing with regular spaces, I have long been curious to know whether certain results proved there could be used to obtain the well known theorem that a separable (Hausdorff) space is metrizable if (and only if) it is regular. Since a positive answer to the question thus posed may have some interest from a methodological point of view, I communicate a demonstration here. The essential step in this demonstration even has some intrinsic interest, consisting as it does in the proof of new facts about dissection-spaces and the related maps. However, as a proof of the metrizability theorem this discussion is not as simple or as direct as the now classical proof of Tychonoff and Urysohn—which, it may be recalled, consists in showing, first, that a separable regular space is normal[1] and, second, that a separable normal space is metrizable.[2]

As a direct corollary of theorems established in our paper cited above, we may state the following result.

THEOREM. *If $\mathfrak{R}$ is a separable regular space, then $\mathfrak{R}$ has a map $m(\mathfrak{R}, \mathfrak{S}, \mathfrak{X})$ where $\mathfrak{X}$ is a continuous family of disjoint closed sets in a compact metric space $\mathfrak{S}$, which may be taken as a closed subset of the Cantor discontinuum; in other words, $\mathfrak{R}$ is topologically equivalent to the space obtained by introducing the "weak" topology in $\mathfrak{X}$.*

Theorems 26 and 69 of our paper show that the desired map can be constructed with $\mathfrak{S}$ taken to be the Boolean space representing the countable Boolean algebra generated by an arbitrarily chosen countable basis for $\mathfrak{R}$; but Theorems 1, 10, and 13 show that the space $\mathfrak{S}$ is topologically equivalent to a closed subspace of the Cantor discontinuum.

We shall now establish the following result.

THEOREM. *Let $\mathfrak{X}$ be a continuous family[3] of mutually disjoint, non-void, compact subsets $\mathfrak{x}$ in a metric space $\mathfrak{S}$. Then the space obtained by*

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[1] Tychonoff, Math. Ann. vol. 95 (1925–1926) pp. 139–142.

[2] Urysohn, Math. Ann. vol. 94 (1925) pp. 309–315.

[3] In the terminology of R. L. Moore, an upper semi-continuous collection.