A PROPERTY OF DERIVATIVES

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If a real function defined over a closed interval $[a, b]$ is differentiable at each point of the interval, it is well known that its derivative possesses the Darboux property: if $f'(c) < \xi < f'(d)$, then there is a point e between c and d with $f'(e) = \xi$.

Now let α, β, with $\alpha < \beta$, be any two fixed reals, and consider the set $E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}$. It is easily seen, as a consequence of the Darboux property, that any such set $E(\alpha, \beta)$ must contain a continuum of points, unless it is empty. The question of the measure of $E(\alpha, \beta)$ does not seem to be covered in the literature, except in the case in which the given interval is either $(-\infty, \beta)$ or $(\alpha, +\infty)$. We prove that any such set $E(\alpha, \beta)$ is either empty or of positive measure.

We remark that this result cannot be deduced from the Darboux property alone; Lebesgue exhibited a function[1] which possesses that property without satisfying the measure condition. Another example is the following. Let C be the Cantor closed nondense set of measure zero and power c in the unit interval, and let $\{T_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of linear transformations such that the sets $T_n(C)$ are disjoint, and such that any sub-interval of $[0, 1]$ contains some $T_n(C)$. We take T_1 to be the identity. Let the function $g(x)$ be defined on C in such a way as to assume all values from zero to one inclusive; on $T_n(C)$ let $g(x) = g(T_n^{-1}(x))$. On all remaining points of the unit interval set $g(x) = 0$. It is clear that this function g possesses the Darboux property, but that the set $E\{x/1/2 < g(x) < 1\}$ will be nonvoid and of measure zero.

THEOREM. *If $f(x)$ is real and everywhere differentiable in the closed interval $[a, b]$, then for any two reals α, β $(\alpha < \beta)$, the set*

$$E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}$$

is empty or of positive measure.

PROOF. We start with the following known result:[2] if a continuous function $f(x)$ is differentiable in the interval $[a, b]$, with the possible exception of a denumerable set of points x, and if $f'(x)$ is non-negative almost everywhere, then $f(x)$ is nondecreasing. It follows that if $f'(x)$ exists for all x in $[a, b]$, and $f'(x) \geqq \lambda$ [or $f'(x) \leqq \mu$] for almost all x,

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[1] Lebesgue, *Leçons sur l'intégration*, 2d ed., Paris, 1928, p. 97.

[2] Saks, *Théorie de l'intégrale*, Warsaw, 1933, p. 141.