NOTE ON MULTIPLY-INFINITE SERIES

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Let $\sum a_{(i)}$ be a k-tuply infinite series, where the subscript (i) stands for the set of k indices $i_1, \cdots, i_k$ and where for each $s=1, \cdots, k$ the index i_s ranges from 0 to ∞. If $\sum b_{(i)}$ is a second such series we can determine from them a third series $\sum c_{(i)}$ called the Cauchy-product series, defined by

$$c_{(i)} = \sum a_{(j)} b_{(n)}, \tag{1}$$

where the sum is over all indices for which, simultaneously,

$$j_s + n_s = i_s, \qquad s = 1, \cdots, k. \tag{2}$$

Series $\sum c_{(i)}$ is obtained by formally multiplying the two power series $\sum a_{(i)} t_1^\alpha \cdots t_k^\lambda$, $\sum b_{(i)} t_1^\alpha \cdots t_k^\lambda$ $(\alpha \equiv i_1, \cdots, \lambda \equiv i_k)$, then setting $t_s = 1$, $s=1, \cdots, k$.

Given that $\sum a_{(i)}$, $\sum b_{(i)}$ converge to A, B respectively, it is natural to ask if $\sum c_{(i)}$ possesses the *Cauchy-product property*, that is, if it converges and to the sum $C=AB$; and if this is not always true, under what further conditions it will be true. It is the purpose of this note to give an answer to this question. All convergence is to be in the sense of Pringsheim (except in the concluding remark).

For simply-infinite series (hence also for multiple series) mere convergence of the series $\sum a_{(i)}$, $\sum b_{(i)}$ is known to be insufficient to insure that $\sum c_{(i)}$ will converge to the right sum. A theorem of Mertens states however that if both given series converge, one of them *absolutely*, then the Cauchy-product property holds.

Like a number of other properties of simple series, the Mertens theorem does not go over unrestrictedly to multiple series that are Pringsheim convergent. We shall show this by an example, after which we shall find suitable restrictions that will restore the Cauchy-product property.

Example. Let $\sum a_{ij}$, $\sum b_{ij}$ be the following double series: $\sum a_{ij}$ is the absolutely convergent series whose first column has the elements

$$1, \frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}, \cdots,$$

while all other terms are zero; $\sum b_{ij}$ is the convergent series whose

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