A CHARACTERIZATION OF SEMI-SIMPLE RINGS WITH THE DESCENDING CHAIN CONDITION

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H. Weyl[1] has defined a semi-simple algebra (of finite rank) to be an algebra which admits a faithful semi-simple linear representation. Now, algebras are rings with a field of operators; Artin and others[2] have shown that the theory of semi-simple algebras can be generalized to a theory of semi-simple rings (without the field of operators) provided we replace the condition of finite rank by suitable finiteness conditions. (Both the ascending and descending chain conditions were assumed, but it was later shown that the descending chain condition was sufficient.[3]) The notion of semi-simplicity is defined by the condition that the radical reduces to $\{0\}$, there being several equivalent definitions of the radical. We introduce another one below.

The question now arises whether the Weyl definition could not be extended to the case of rings. To do this, we must extend to an arbitrary ring the notion of a linear representation of an algebra. This can be done by replacing the consideration of the algebra of matrices by the more general notion of the ring of endomorphisms of an abelian group: a *representation* of a ring A will be a homomorphism ρ of A into the ring of endomorphisms of an additive group $\mathfrak{M}$. Let such a representation be given; we can define a law of composition, $(a, m) \to am$, between elements of A and of $\mathfrak{M}$ by writing $am = \{\rho(a)\}(m)$. The composite object formed by $\mathfrak{M}$ and this law of composition is called an *A-module.* A *sub-module* of an *A-module* $\mathfrak{M}$ is a subset $\mathfrak{N}$ of $\mathfrak{M}$ which is a subgroup of the additive group of $\mathfrak{M}$ and is such that $A\mathfrak{N} \subset \mathfrak{N}$. ($A\mathfrak{N}$ is defined to be the set of all finite sums $\sum_i a_i m_i$, $a_i \in A$, $m_i \in \mathfrak{N}$.) A *homomorphism* of an A-module $\mathfrak{M}$ into an A-module $\mathfrak{M}'$ is a homomorphism h of the additive group of $\mathfrak{M}$ into the additive group of $\mathfrak{M}'$ which is such that $h(am) = ah(m)$ for all $a \in A$, $m \in \mathfrak{M}$.

An A-module $\mathfrak{M}$ is said to be *simple* if its only submodules are $\{0\}$ and itself. Concerning simple modules, we have the well known lemma:

SCHUR'S LEMMA. *A homomorphism h of a simple A-module $\mathfrak{M}$ into*

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[1] H. Weyl, *The classical groups*, Princeton University Press, 1939, p. 85.

[2] E. Artin, Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1927) p. 251; J. Levitzki, Compositio Math. vol. 5 (1937) p. 392.

[3] R. Brauer, Bull. Amer. Math. Soc. vol. 48 (1942) p. 752; C. Hopkins, Duke Math. J. vol. 4 (1938) p. 664; J. Levitzki, Compositio Math. vol. 7 (1939) p. 214.