THE REPRESENTATION OF e^{-x^λ} AS A LAPLACE INTEGRAL

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According to a theorem of Bochner [1, p. 498][1] the function e^{-x^λ}, for any fixed value of λ in $0<\lambda<1$, is completely monotonic and admits a unique representation

$$e^{-x^\lambda} = \int_0^\infty e^{-xt} d\alpha_\lambda(t), \qquad 0 \leqq x < \infty,$$

where $\alpha_\lambda(t)$ is bounded and increasing. It follows further from a criterion of Hille and Tamarkin [2, p. 903] that the function also has the form

$$e^{-x^\lambda} = \int_0^\infty e^{-xt}\phi_\lambda(t)\, dt. \tag{1}$$

One can conclude therefore, since $\alpha_\lambda'(t)=\phi_\lambda(t)$, that $\phi_\lambda(t)$ is positive almost everywhere and that

$$\int_0^\infty \phi_\lambda(t) dt < \infty.$$

For this last integral is the total variation of $\alpha_\lambda(t)$, suitably normalized.

Further information concerning $\phi_\lambda(t)$ may be derived from some general results of Post. Let γ be the contour

$$\frac{x}{a} + \frac{|y|}{b} = 1$$

where a and b are fixed and positive; their precise values are a matter of indifference. The principal branch of e^{-z^λ} is holomorphic in the sector to the right of γ, and is moreover of zero type there since $0<\lambda<1$. If e^{-z^λ} is denoted by $f(z)$, the theory of Post [3, p. 730] shows that the limit

$$L[f;t] = \lim_{kh\to t, h\to 0+} \frac{(-1)^k}{k!} h^{-k-1} f^{(k)}\left(\frac{1}{h}\right) = \frac{1}{2\pi i}\int_\gamma e^{zt}e^{-z^\lambda} dz$$

exists for all $t>0$; γ must be traced so that the origin is at the left. But according to the Post-Widder inversion theorem [4, p. 288] $L[f;t]$ is the inverse Laplace transform of e^{-x^λ}, and so must be equal

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[1] Numbers in brackets refer to the references cited at the end of the paper.