

A SECOND NOTE ON WEAK DIFFERENTIABILITY OF PETTIS INTEGRALS

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In a recent paper¹ the author proved that if Ω is any compact metric space containing non-denumerably many points and $C(\Omega)$ is the Banach space of all continuous functionals over Ω , then there is a Pettis integrable function from the unit interval to $C(\Omega)$ whose integral fails to be weakly differentiable on a set of positive measure. The purpose of this note is to obtain the same result, assuming only that Ω contains infinitely many points. This leads to a certain necessary and sufficient condition for weak differentiability of Pettis integrals.

The author's previous paper (cited above) will be referred to hereafter as Note I. It will be assumed that the reader is familiar with the notation, terminology, and results of that paper.

Let B be the non-dense perfect set described in Note I, and let \bar{B} be its complement. Let the intervals of \bar{B} be arranged in a sequence I_1, I_2, I_3, \dots in such a way that if the order of I_j is greater than the order of I_k , then $j > k$. For each positive integer k , let $n(k)$ be the order of I_k . As in Note I, we now define a function $\phi(x, t)$ over the unit square so that for each x , $\phi(x, t)$ is continuous in t . This will serve to define a function from the unit interval to the space C .

For $x \in B$, let $\phi(x, t) = 0$; for $x \in I_1$, let $\phi(x, t) = 0$; for $x \in I_k$ and $t = 1/k$ ($k = 2, 3, 4, \dots$), let $\phi(x, t) = 2^{2n(k)}/n(k)$; for $x \in I_k$ and $t = 1/(k \pm 1/2)$ ($k = 2, 3, 4, \dots$), let $\phi(x, t) = 0$; for $t = 0$ or 1 , let $\phi(x, t) = 0$. Now for each x , let $\phi(x, t)$ be extended linearly as a function of t between successive points already determined. We now denote by $\phi(x)$ the function whose values are the elements of C determined by $\phi(x, t)$.

THEOREM 1. $\phi(x)$ is Pettis integrable. Its integral is the element of C

$$\Phi_E(t) = \int_E \phi(x, t) dx.$$

That $\Phi_E(t)$ is defined and continuous in t is obvious. For $t = 1/k$, $\Phi_E(t) = |E \cdot I_k| 2^{2n(k)}/n(k) = |E \cdot I_k| / |I_k| n(k) \leq 1/n(k)$; for

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¹ M. E. Munroe, *A note on weak differentiability of Pettis integrals*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 167-174.