ON THE EXTENSION OF HOMEOMORPHISMS ON THE INTERIOR OF A TWO CELL

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The subject under discussion in this paper is the study of the existence and properties of extensions of homeomorphisms of the interior I of a two cell with boundary C onto a plane bounded region. Particular emphasis will be placed on the action of the extension on C. Application of the topological results will then be made to conformal maps on the interior of the unit circle.

The hypothesis that $f(I)=R$ is a homeomorphism of the interior I of a two cell with boundary C onto a plane bounded region R with boundary $F(R)$ will be assumed throughout the paper. The usual terminology of transformation theory will be used: the transformation $g(A)=B$ is said to be light if each $f^{-1}(x)$, $x\in B$, is totally disconnected, and non-alternating if for each x, $y\in B$, $f^{-1}(x)$ does not separate $f^{-1}(y)$.[1]

1. Action of extensions on the boundary.

THEOREM 1. *Suppose f is uniformly continuous. Then there exists a continuous extension g of f such that $g(\bar{I})=\bar{R}$ and $g=f$ on I. Moreover $g(C)=F(R)$ is a non-alternating transformation.*

PROOF. The existence of the extension is well known, since f is uniformly continuous. Moreover $g(C)=F(R)$. To prove this, we notice that $g(\bar{I})$ is compact and must contain $\bar{R}$. Since $g(I)=R$, then $g(C)\supset F(R)$. Suppose $g(C)\neq F(R)$; then there is a point $x\in C$ such that $g(x)\in R$. Let $(x_i)\to x$, $x_i\in I$; then $(f(x_i))\to g(x)$. Since $g(x)\in R$, then $(x_i)\to f^{-1}g(x)\in I$. This is a contradiction and $g(C)=F(R)$.

Suppose $g(C)=F(R)$ is not non-alternating; then there exist points $x_1, x_2, y_1, y_2\in C$ such that $g(x_1)=g(x_2)$, $g(y_1)=g(y_2)$, $g(x_1)\neq g(y_1)$, and x_1+x_2 separates y_1+y_2 on C. Let A_1 and A_2 be interiors of arcs x_1x_2 and y_1y_2 respectively, where $x_1x_2\subset I+x_1+x_2$, $y_1y_2\subset I+y_1+y_2$, $x_1x_2\cdot y_1y_2=p=A_1A_2$. Both $g(x_1x_2)$ and $g(y_1y_2)$ are simple closed curves and $g(x_1x_2)\cdot g(y_1y_2)=f(p)$. Moreover points of $g(y_1y_2)$ are contained both in the interior and exterior of $g(x_1x_2)$. For A_1 separates A_2 into two parts, one in each component of $I-A_1$; then $f(A_1)$ separates $f(A_2)$ into two parts, one in each component of $R-f(A_1)$. But one compo-

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[1] See G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942, pp. 127–129, 138–140, 165–170 for properties of non-alternating maps.