TAYLOR'S SERIES AND APPROXIMATION TO ANALYTIC FUNCTIONS

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Taylor's series, as the simplest expansion in terms of rational functions of an arbitrary analytic function of a complex variable, has shown itself extremely useful as a guide to other such expansions. Taylor's series itself suggests other expansions and their properties, for instance series of polynomials defined by interpolation or best approximation, and is frequently a special case or a limiting case of these expansions.

The object of this address is to indicate that Taylor's series is also useful both as a guide and as a tool in the study of still other general expansions of analytic functions, namely where each approximating function is assumed merely analytic and bounded in a given region, and where the function approximated is assumed merely analytic in a closed subregion. We shall establish certain results in detail, and later indicate some new unpublished results and some open problems.

For convenient reference we mention some properties of the Taylor development

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \tag{1}$$

of a function $f(z)$ analytic throughout the circle $|z| < R$ (>1) but analytic throughout no larger concentric circle. If we set

$$S_n(z) = a_0 + a_1 z + \cdots + a_n z^n, \tag{2}$$

then we have

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1/R, \tag{3}$$

from which we find

$$\limsup_{n\to\infty} [\max |f(z) - S_n(z)|, \text{ for } |z| = r]^{1/n} = r/R, \qquad r < R, \tag{4}$$

$$\limsup_{n\to\infty} [\max |S_n(z)|, \text{ for } |z| = r]^{1/n} = r/R, \qquad r > R. \tag{5}$$

Let us now consider the following problem, with relation to this same function $f(z)$. Let $R_0 > R$ be fixed, and let $M > 0$ be chosen; we shall later allow M to become infinite. Denote by $f_M(z)$ the (or a)

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