SOME REMARKS ABOUT ADDITIVE AND MULTIPLICATIVE FUNCTIONS

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The present paper contains some results about the classical multiplicative functions $\phi(n)$, $\sigma(n)$ and also about general additive and multiplicative functions.

(1) It is well known that $n/\phi(n)$ and $\sigma(n)/n$ have a distribution function.[1] Denote these functions by $f_1(x)$ and $f_2(x)$. ($f_1(x)$ denotes the density of integers for which $n/\phi(n) \leqq x$.) It is known that both $f_1(x)$ and $f_2(x)$ are strictly incrtasing and purely singular.[1] We propose to investigate $f_1(x)$ and $f_2(x)$; we shall give details only in case of $f_1(x)$. First we prove the following theorem.

THEOREM 1. *We have for every* ϵ *and sufficiently large* x

$$(1) \quad \exp(-\exp[(1+\epsilon)ax]) < 1 - f_1(x) < \exp(-\exp[(1-\epsilon)ax])$$

where $a=\exp(-\gamma)$, γ *Euler's constant.*

We shall prove a stronger result. Put $A_r = \prod_{i=1}^{r} p_i$, p_i consecutive primes. Define A_k by $A_k/\phi(A_k) \geqq x > A_{k-1}/\phi(A_{k-1})$. Then we have

$$(2) \qquad 1/A_k < 1 - f_1(x) < 1/A_k^{1-\epsilon}.$$

First of all it is easy to see that Theorem 1 follows from (2), since from the prime number theorem we easily obtain that $\log\log A_k = (1+o(1))ax$, which shows that (1) follows from (2).

(2) means that the density of integers with $\phi(n) \leqq (1/x)n$ is between $1/A_k$ and $1/A_k^{1-\epsilon}$.

We evidently have for every $n \equiv 0 \pmod{A_k}$, $n/\phi(n) \geqq x$, which proves

$$1/A_k \leqq 1 - f_1(x).$$

To get rid of the equality sign, it will be sufficient to observe that there exist integers u with $u/\phi(u) \geqq x$, $(u, A_k) = 1$, and that the density of the integers $n \equiv 0 \pmod{u}$, $n \not\equiv 0 \pmod{A_k}$ is positive. This proves the first part of (2). The proof of the second part will be much harder. We split the integers satisfying $n/\phi(n) \geqq x$ into two classes. In the first class are the integers which have more than $[(1-\epsilon_1)k] = r$ prime factors not greater than Bp_k, where $B = B(\epsilon_1)$ is a large number. In

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[1] These results are due to Schönberg and Davenport. For a more general result see P. Erdös, J. London Math. Soc. vol. 13 (1938) pp. 119–127.