ON THE COMMUTATIVITY OF CERTAIN RINGS

ALEXANDRA FORSYTHE AND NEAL H. McCOY

1. **Introduction.** In his fundamental paper [5][1] on Boolean rings, Stone showed that a Boolean ring, that is, a ring R with the property that $a^2=a$ for every element a of R, is necessarily commutative. Recently, Kaplansky [3] announced that this result can be extended and that, under certain rather strong conditions on the positive integer n, a ring R is commutative if $a^n=a$ for every element a of R. Furthermore, Jacobson [2] has now shown that this is true without restriction on n. In fact, he has established the following more general result:

THEOREM 1 (JACOBSON). *If for each element a of a ring R there exists an integer $n(a)>1$, depending on a, such that $a^{n(a)}=a$, then R is commutative.*

A simple calculation [2, p. 702] shows that every element of a ring satisfying the hypothesis of this theorem has finite additive order. Thus if a *division ring* satisfies the hypothesis, its prime field is necessarily finite. Theorem 8 of [2] then furnishes a short and elegant proof of Theorem 1 for the case of division rings. The proof of the theorem can then be completed by obtaining a "reduction to division rings," that is, a proof that the theorem is true for all rings if it is true for all division rings. Jacobson accomplishes this by use of some rather deep results on algebraic algebras. The principal purpose of the present note is to present a simple reduction to division rings which was obtained independently by the present authors. This, coupled with Jacobson's proof of the result for division rings, furnishes a short and simple proof of Theorem 1.

We shall give, in §3, an entirely elementary proof of this theorem for the special case in which R is of prime characteristic p and $a^p=a$ for every element a of R. Such a ring is a *p-ring* [4], which is perhaps the simplest and most natural generalization of a Boolean ring.

2. **The reduction to division rings.** We shall, in fact, treat a somewhat more general case which may be of some interest in itself.

We recall that, according to Birkhoff [1], a ring R is *subdirectly irreducible* if the intersection of all nonzero two-sided ideals in R is a nonzero ideal. The following fundamental theorem is essential for our purpose:

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[1] Numbers in brackets refer to the references cited at the end of the paper.