ABSOLUTE RETRACTS IN GROUP THEORY

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The subgroup R of the group G has been termed a retract[1] of the group G whenever there exists an idempotent endomorphism of G which maps G upon R. This definition is in strict analogy to the topological concept of retract. Thus one may be tempted to define absolute retracts in like similarity to topological usage. However, we shall prove in the course of the present note that the identity is the only group which is a retract of every containing group. Consequently only modifications of the topological concept will be useful, and we shall show that each of the following classes of groups may in a certain sense lay claim to the title of absolute retract: the complete groups, the abelian groups the orders of whose elements are finite and square free, and the free groups.

The following definition of the concept of retract is equivalent to the one given above, but it will be a little bit easier to handle: The group R is a retract of the group G if R is a subgroup of G, and if there exists an endomorphism ϵ of G with the following properties:

$$G^{\epsilon} \leqq R \quad \text{and} \quad r^{\epsilon} = r$$

for r in R.

Complete groups (Carmichael [2, p. 81]) have been defined as groups with the following properties: each of their automorphisms is an inner automorphism and their center consists of the identity only. Thus groups are complete if they are "essentially" identical with their group of automorphisms.

THEOREM 1. *The group G is complete if, and only if, it meets the following requirement*:

(*) *If G is a normal subgroup of the group E, then G is a retract of E.*

PROOF. Assume first the validity of condition (*). Consider the automorphism α of G. Then there exists (Zassenhaus [1, pp. 93, 94]) one and essentially only one group A which is obtained by adjoining to G an element t subject to the relations

$$t^{-1}gt = g^{\alpha}$$

for g in G. Clearly G is a normal subgroup of A and A/G is an infinite

Presented to the Society, April 27, 1946; received by the editors January 30, 1946.

[1] See Baer [2, chap. II, §3] for elementary properties of retracts. Many applications of the concept of retract may be found there too. Numbers in brackets refer to the Bibliography at the end of the paper.