ON A PROBLEM OF KUROSCH AND JACOBSON

IRVING KAPLANSKY

1. **Introduction.** Let A be an algebraic algebra over a field K, that is, an algebra over K each of whose elements satisfies a polynomial equation with coefficients in K. In analogy to Burnside's problem for groups, Kurosch[1] has raised the following question: if A is finitely generated, does it necessarily have a finite basis? Jacobson[2] studied the question for the case where the elements of A are of bounded degree, and reduced it to the consideration of certain specific nil algebras defined as follows: $A(r, n) = F(r) - I(r, n)$, where $F(r)$ is the free algebra generated over K by indeterminates $u_1, \cdots, u_r$, and $I(r, n)$ is the (two-sided) ideal generated by all nth powers in $F(r)$. In this note we shall prove the following theorem.

THEOREM 1. *If K has at least n elements, $A(r, n)$ has a finite basis.*

Thus Kurosch's question for algebraic algebras of bounded degree receives an affirmative answer if K is large enough, and in particular if it is infinite.

In §3, by a different method suggested by Kurosch's treatment of $n=3$, we prove that $A(r, 4)$ has a finite basis over $GF(3)$. In §4 we discuss a special case of another question proposed by Jacobson: if the dimension $d(r, n)$ of $A(r, n)$ is finite, what is its precise value? We show that $d(2, 3)$ may be equal to 16 or 17, depending on K.

2. **Proof of Theorem 1.** Throughout this section we shall assume that the coefficient field K has at least n elements.

The algebra $F(r)$ consists of all (noncommutative) polynomials in the u's with coefficients in K: that is, linear combinations of terms $u_i u_j u_k \cdots$ which we shall call monomials. The degree of a monomial is the number of u's it contains, and a polynomial is homogeneous if its monomials all have the same degree. We now prove the following lemma.

LEMMA 1. *The ideal $I(r, n)$ has a basis of homogeneous polynomials.*

PROOF. Specifically, $I = I(r, n)$ has a basis consisting of all

Received by the editors February 13, 1946.

[1] *Ringtheoretische Probleme die mit dem Burnsideschen Problem über periodische Gruppen in Zusammenhang stehen*, Bull. Acad. Sci. URSS. Ser. Math. vol. 5 (1941) pp. 233–240.

[2] *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. (2) vol. 46 (1945) pp. 695–707.