TOPOLOGICAL GROUPS IN WHICH MULTIPLICATION OF ONE SIDE IS DIFFERENTIABLE

I. E. SEGAL

The present note is concerned with the problem of determining the formally weakest conditions on a group which can be proved to result in the group being a Lie group. We show essentially that it is sufficient for this purpose to require that the group be a (real) manifold of class C^1 in which right multiplication is of class C^1 (our result is slightly stronger—Theorem 2 is the precise statement). The weakest previously existing condition is contained in work of P. A. Smith [2],[1] and required, in addition to the preceding condition, that left multiplication satisfy a Lipschitz condition.[2]

We should point out that our result is obtained by combining previous work on Lie groups with a theorem which we prove independently of previous work. Specifically, we first prove that if right multiplication is of class C^1 on a group G which is a manifold of class C^1, then it follows that left multiplication is of class C^1 (Theorem 1). We then deduce that G is a Lie group either from the result of Smith quoted above, or from the result of G. Birkhoff [1] that if both left and right multiplication are of class C^1, then G is a Lie group. We do not introduce any kind of canonical parameters, and, so far as we know, neither the work of Smith nor that of Birkhoff can be simplified by the use of our results.

Our method is novel in that it utilizes the existence of Haar measure and operates in the large. (Smith and Birkhoff were both concerned exclusively with the theory in the small, a standpoint incompatible with the use of Haar measure.) In this way it is shown that a wide collection of functions of x of class C^1, x being a variable group element, are also of class C^1 as functions of x^{-1}, and most of the proof is occupied with showing that this collection is sufficiently wide to allow the conclusion to be drawn that x^{-1} is of class C^1 as a function of x. It follows that left multiplication is of class C^1. Only minor use is made of the theory of differential equations.

We are indebted to D. Montgomery for highly suggestive conversations about the foundations of the theory of Lie groups.

The following theorem is valid for functions and manifolds of class

Presented to the Society, February 23, 1946; received by the editors January 19, 1946.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Smith had suggested that the Lipschitz condition might be inessential.