TOEPLITZ METHODS WHICH SUM A GIVEN SEQUENCE

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The following note arose out of discussions of a paper by Agnew, but is, however, self-contained.

THEOREM. *Let $\{x_n\}$ be a bounded divergent sequence. Suppose that $\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$. Then $\{y_n\}$ is of the form $\{cx_n+a_n\}$ where $\{a_n\}$ is convergent.*

PROOF. For typographical convenience we shall often write $x(n)$ for x_n, and so on. Let $\{x(n_k)\}$, $k=1, 2, \cdots$, be any convergent subsequence of $\{x_n\}$. Then $\{x_n\}$ is summable by the matrix $(a(n, k))$ where $a(n, k)=1$ for $n=n_k$ and $a(n, k)=0$ for $n\neq n_k$. Hence $\{y(n_k)\}$ is also convergent.

Let $\{n_k'\}$ and $\{n_k''\}$ be sequences of integers such that $n_k'\neq n_k''$ for all k and

$$\lim_{k\to\infty} x(n_k') = A, \qquad \lim_{k\to\infty} x(n_k'') = B, \qquad A \neq B.$$

These sequences $\{n_k'\}$ and $\{n_k''\}$ will be held fixed throughout the rest of the argument. Then the sequences $\{y(n_k')\}$ and $\{y(n_k'')\}$ are also convergent, say to α and β respectively. Let $\{x(n_k)\}$ be an arbitrary convergent subsequence of $\{x_n\}$ with the limit C. Let λ and μ be determined by the equations

$$\lambda + \mu = 1, \qquad \lambda A + \mu B = C.$$

Then the matrix $(b(n, k))$ with

$$b(n, k) = \begin{cases} \lambda, & n = n_k', \quad k \text{ even}, \\ \mu, & n = n_k'', \quad k \text{ even}, \\ 1, & n = n_k, \quad k \text{ odd}, \\ 0, & \text{for all other values of } n \text{ and } k, \end{cases}$$

sums $\{x_n\}$ to the limit C. Hence it also sums $\{y_n\}$, that is

$$\lim_{k\to\infty} y(n_k) = \lim_{k\to\infty} (\lambda y(n_k') + \mu y(n_k'')) = \lambda\alpha + \mu\beta.$$

Note that the numbers λ and μ depend only on C and not on the particular subsequence $\{x(n_k)\}$ converging to C, and hence $\lim_{k\to\infty} y(n_k)$

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