SOME PROPERTIES OF ABSOLUTELY MONOTONIC FUNCTIONS

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In this note we collect several fragmentary results which were obtained as by-products of another investigation. They are rather loosely connected with each other, but still may be of some interest.

We recall that a function $f(x_1, \cdots, x_k)$ is said to be absolutely monotonic in a set D if f and all its partial derivatives exist and are non-negative in D. If D is of the form $0 \leqq x_i < a_i$, $i=1, \cdots, k$, then a necessary and sufficient condition that f be absolutely monotonic in D is that it can be expanded in a power series in $x_1, \cdots, x_k$ with non-negative coefficients converging in D. (The well known theorem of Bernstein [1][1] for the case $k=1$ can be extended in a trivial manner.)

THEOREM 1. *If $f(x)$ is absolutely monotonic in $0 \leqq x < a$, and if $0 \leqq x_1, x_2, \cdots, x_n < a$, and if $L(x)$ is the Lagrange interpolation polynomial of $f(x)$ at the points $x_1, \cdots, x_n$, then*

$$g(x) = \frac{f(x) - L(x)}{\omega(x)}, \qquad \omega(x) = (x - x_1) \cdots (x - x_n),$$

is an absolutely monotonic function of $x, x_1, \cdots, x_n$ in the range $0 \leqq x, x_1, \cdots, x_n < a$.

PROOF. The function $g(x)$ can be expressed as a divided difference of $f(x)$ (see for example, Milne-Thompson [2]):

$$g(x) = [x x_1 \cdots x_n],$$

where

$$[x x_1] = \frac{f(x) - f(x_1)}{x - x_1},$$

and

$$[x x_1 \cdots x_k] = \frac{[x x_1 \cdots x_{k-1}] - [x_k x_1 \cdots x_{k-1}]}{x - x_k}, \qquad k = 2, \cdots, n.$$

It is sufficient, then, to show that if $f(x)$ is absolutely monotonic in $0 \leqq x < a$ then

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.