PROPERTIES EQUIVALENT TO THE COMPLETENESS OF $\{e^{-t}t^{\lambda_n}\}$

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We are concerned with the following three properties which may be possessed by an increasing sequence $\{\lambda_n\}$ of positive integers.

(A) If $\{a_n\}$ is a sequence of complex numbers such that, for some β, $a_n = O(n^\beta)$ and $\Delta^{\lambda_n} a_0 = 0$ $(n = 1, 2, \cdots)$, a_n is a polynomial in n; here

$$\Delta^n a_0 = \sum_{k=0}^{n} (-1)^k C_{n,k} a_k.$$

(B) The set $\{t^{\lambda_n} e^{-ct}\}$ is complete $L^2(0, \infty)$; that is,

$$\int_0^\infty t^{\lambda_n} e^{-ct} \phi(t) dt = 0 \qquad (n = 1, 2, \cdots; \phi \in L^2)$$

implies $\phi(t) = 0$ almost everywhere.[1]

(C) If $f(z)$ is regular and $O(|z|^\alpha)$ for some α in the half-plane $x > -\epsilon$, $\epsilon > 0$, and $f^{(\lambda_n)}(0) = 0$ $(n = 1, 2, \cdots)$, $f(z)$ is a polynomial.[2]

W. H. J. Fuchs [3][3] showed that (A) and (B) are equivalent. We shall give a somewhat simpler proof, and show in addition that (C) is equivalent to (A) and (B).

Fuchs showed that (A) is true if $n(r) \geqq r/2 - \gamma$ for some constant γ, where $n(r)$ is the number of $\lambda_n \leqq r$. R. P. Agnew discovered independently [1] that (A) is true if $\lambda_n = 2n$; a simplified proof given by Pollard [5] was the starting point of this note. Boas [2] has shown by other methods that it is enough to have $n(r) \geqq r/2 - r\delta(r)$, where $\int^\infty r^{-1}\delta(r)dr$ converges and $\delta(r)$ satisfies some mild auxiliary conditions. (Fuchs, in a paper [3a] which appeared while this note was in the press, has shown that a necessary and sufficient condition for (A) is that $\int^\infty r^{-2}\psi(r)dr$ diverges, where $\log \psi(r) = 2\sum_{\lambda_n \leqq r} \lambda_n^{-1}$.)

Let $P(\lambda_n)$ mean that $\{\lambda_n\}$ has property (P); $P(\lambda_n - N)$, that the sequence $\{\lambda_n - N\}$ has (P), where $\lambda_n - N$ is replaced by 0 if $\lambda_n < N$. Our line of reasoning is schematically as follows: $A(\lambda_n) \to B(\lambda_n) \to C(\lambda_n + N) \to A(\lambda_n + N) \to A(\lambda_n - N) \to B(\lambda_n - N) \to C(\lambda_n) \to A(\lambda_n)$. It would be more direct to use $B(\lambda_n) \to B(\lambda_n - N)$; this can be quoted from the

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[1] Replacing ct by t, we see that (B) is independent of c.

[2] (C) thus concerns the analytic continuation of a function defined by a lacunary power series $\sum c_n z^{\mu_n}$, where $\{\mu_n\}$ is the sequence of positive integers complementary to $\{\lambda_n\}$.

[3] Numbers in brackets refer to the references at the end of the paper.