AREOLAR MONOGENIC FUNCTIONS

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1. Introduction. There have been several modifications of the definition of the derivative of a complex function of a complex variable which lead to theories of non-analytic functions. These generalizations were initiated by Riemann (1851) and Picard (1892) and followed with others by Pompeiu, Kasner and Cioranescu. The general derivatives of Riemann and Cioranescu depend on direction and have an infinite set of values at a given point, hence Kasner gave to the class of non-analytic functions considered the name polygenic functions to distinguish them from classical analytic, monogenic functions.[1]

The conditions for classical monogenity have been much reduced by Looman-Menchoff [7, pp. 9–16; 9, pp. 198–201].[2] We shall similarly reduce the restrictions for the existence of the Cioranescu single-valued areolar derivative and show that under those reduced conditions the real and imaginary parts of the areolar monogenic function are biharmonic. Finally the class of areolar monogenic functions so determined will be simply characterized in terms of the Pompeiu derivative.

2. The Cioranescu and Pompeiu derivatives. Let $f(z)=f(x, y)=u(x, y)+iv(x, y)$ be defined in a domain D of the complex variable $z=x+iy$. Construct a rectangle in D at a point z of D whose vertices in positive order are z, z_1, z', z_2. If z is taken as the pole of a polar coordinate system (ρ, ϕ) then $z_1-z=\rho_1e^{i\phi}$, $z_2-z=\rho_2e^{i(\phi+\pi/2)}$ and $z'-z=(\rho_1{}^2+\rho_2{}^2)^{1/2}e^{i(\phi+\alpha)}$ where $\alpha=\tan^{-1}\rho_2/\rho_1$. We now form the quotient

$$(2.1)\qquad \Delta^2f(z)=\frac{f(z')-f(z_1)-f(z_2)+f(z)}{(z_1-z)(z_2-z)}$$

and consider the limit of $\Delta^2f(z)$ as ρ_1 and ρ_2 approach zero with ϕ held

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[1] See E. R. Hedrick, *Non-analytic functions of a complex variable*, Bull. Amer. Math. Soc. vol. 39 (1933) pp. 75–96 for an extensive bibliography. The author is indebted to the referee for the following observation. "Calugareano studied the second derivative of a polygenic function for only one rectilinear path of approach; Nicolesco studied it for any two different rectilinear paths of approach and Cioranescu considered the limit for two mutually perpendicular, rectilinear paths. Kasner and DeCicco have studied the geometry of the second derivative for a general curvilinear path of approach."

[2] Numbers in brackets refer to the references cited at the end of the paper.