ON PROXIMATE ORDERS OF INTEGRAL FUNCTIONS

S. M. SHAH

Let $f(z)$ be an integral function of finite order ρ and let $M(r) = \max_{|z|=r}|f(z)|$. It is possible to find[1] a positive continuous function $\rho(r)$ having the following properties.

(1) $\rho(r)$ is differentiable for $r > r_0$ except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist;

$$\lim_{r\to\infty} \rho(r) = \rho; \tag{2}$$

$$\lim_{r\to\infty} r\rho'(r) \log r = 0; \tag{3}$$

$$\limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho(r)}} = 1. \tag{4}$$

Such a function is called a Lindelöf's proximate order for the integral function $f(z)$. The proof given by Valiron for the existence of proximate orders is based on some rather deep results due to Blumenthal. The object of this note is to give a particularly simple proof of the existence of proximate orders. The proof given here makes no use of the special properties of $M(r)$ and is therefore of wider scope.

Let $\sigma(r) = \log \log M(r)/\log r$. Either (A) $\sigma(r) > \rho$ for a sequence of values of r tending to infinity, or (B) $\sigma(r) \leqq \rho$ for all large r.

In case (A) we define $\phi(r) = \max_{x \geqq r}\{\sigma(x)\}$. Since $\sigma(r)$ is continuous, $\limsup_{r\to\infty} \sigma(r) = \rho$ and $\sigma(r) > \rho$ for a sequence of values of r tending to infinity. Therefore $\phi(r)$ exists. $\phi(r)$ is a nonincreasing function of r.

Let $r_1 > e^{e^e}$ and $\phi(r_1) = \sigma(r_1)$. Such values will exist for a sequence of values of r tending to infinity.

Let $\rho(r_1) = \phi(r_1)$. Let t_1 be the smallest integer not less than $1 + r_1$ such that $\phi(r_1) > \phi(t_1)$, and let $\rho(r) = \rho(r_1) = \phi(r_1)$ for $r_1 < r \leqq t_1$.

Define u_1 as follows:

$$u_1 > t_1,$$

$$\rho(r) = \rho(r_1) - \log\log\log r + \log\log\log t_1 \qquad \text{for } t_1 \leqq r \leqq u_1,$$

$$\rho(r) = \phi(r) \qquad \text{for } r = u_1,$$

but $\rho(r) > \phi(r)$ for $t_1 \leqq r < u_1$. Let r_2 be the smallest value of r for which $r_2 \geqq u_1$ and

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[1] G. Valiron, *Lectures on the general theory of integral functions*, Cambridge, 1923, pp. 64–67.