PROJECTIVITIES WITH FIXED POINTS ON EVERY LINE OF THE PLANE

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The system of fixed elements of a projectivity contains with any two points the line connecting them and with any two lines their intersection. It is, therefore, in its structure very much like a subplane of the plane under consideration; and thus one may expect the structure of the projectivity to be dominated by the structure of the system of its fixed elements, provided this system is not "too small." To substantiate this we propose to investigate in this note a class of projectivities which we term quasi-perspectivities. They are characterized by the property that every line carries a fixed point or, equivalently, that every point is on some fixed line.

Every perspectivity is a quasi-perspectivity, and a quasi-perspectivity is not a perspectivity if, and only if, the system of its fixed elements is a projective subplane. Involutions are quasi-perspectivities too, and if the Theorem of Pappus is valid in the plane under consideration, then every quasi-perspectivity is a perspectivity or an involution. But already in the projective plane over the field of real quaternions there exist quasi-perspectivities which are neither perspectivities nor involutions, and we give a complete survey of the quasi-perspectivities in Desarguesian projective planes.

Our results become particularly striking in the case of finite projective planes. If every line in such a plane carries $n+1$ points, then we may show that there do not exist projectivities possessing exactly n fixed points, that a projectivity is a quasi-perspectivity if, and only if, the number of its fixed points is at least $n+1$, and that it is a perspectivity if, and only if, the number of its fixed points is $n+1$ or $n+2$. If a quasi-perspectivity is not a perspectivity, then $n=i^2$ where $i+1$ is the number of fixed points on a fixed line.

The following notations will be used throughout.

We consider a projective plane Π in which the Theorem of Desargues may or may not hold.[1] If P and Q are two different points in Π, then $P+Q$ is the uniquely determined line passing through P and Q; if h and k are two different lines, then hk is the uniquely determined point in which they meet.

A projectivity ϕ is a $1:1$ and exhaustive correspondence between the

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[1] For a definition of "projective plane" see, for example, Baer [1, p. 138]. Numbers in brackets refer to the Bibliography at the end of the paper.