SOME PROPERTIES OF PARTIAL SUMS OF THE HARMONIC SERIES

PAUL ERDÖS AND IVAN NIVEN

It has been proved that $\sum_{k=m}^{n} k^{-1}$ cannot be an integer[1] for any pair of positive integers m and n. More generally, $\sum_{k=0}^{n}(m+kd)^{-1}$ cannot be an integer.[2] We prove two theorems of a similar nature.

THEOREM 1. *There is only a finite number of integers n for which one or more of the elementary symmetric functions of* $1, 1/2, 1/3, \cdots, 1/n$ *is an integer.*

PROOF. Let $\sum_{k,n}$ denote the kth symmetric function of $1, 1/2, 1/3, \cdots, 1/n$. Since each term of $\sum_{k,n}$ is contained $k!$ times in the expansion of $(1+1/2+\cdots+1/n)^k$, we have, for $k>3\log n$ and n sufficiently large,

$$\sum_{k,n} < \frac{(1+1/2+\cdots+1/n)^k}{k!} < \frac{(1+\log n)^k}{k!} < 1,$$

where the second inequality arises from the usual comparison of $\log n$ with the harmonic series, and the third inequality is implied by the hypothesis $k>3\log n$.

Henceforth we take $k<3\log n$. By a theorem of A. E. Ingham[3] there is a prime between x and $x+x^{5/8}$. This implies that there is a prime p between $1+n/(k+1)$ and n/k for $k<3\log n$ and n sufficiently large. Hence $\sum_{k,n}$ contains the term

$$\frac{1}{p}\cdot\frac{1}{2p}\cdots\frac{1}{kp}=\frac{1}{k!p^k}.$$

Now $(k!, p)=1$ since $k<n/(k+1)$, and hence no other term in $\sum_{k,n}$ has a denominator divisible by p^k. So if $\sum_{k,n}=a/b$, we know that $p^k | b$ and $p \nmid a$, which proves the theorem.

By a similar but more complicated argument we can prove the same

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[1] Cf. Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, Berlin, 1925, chap. 8, p. 159, problem 250.

[2] Cf. T. Nagell, *Eine Eigenschaft gewissen Summen*, Skrifter Oslo, no. 13 (1923) pp. 10–15.

[3] *On the difference between consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) p. 256. This result is actually stronger than necessary for our use here. The classical estimates will suffice.