ON THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL

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The cyclotomic polynomial $F_n(x)$ is defined as the polynomial whose roots are the primitive nth roots of unity. It is well known that

$$F_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)}.$$

For $n<105$ all coefficients of $F_n(x)$ are ± 1 or 0. For $n=105$, the coefficient 2 occurs for the first time. Denote by A_n the greatest coefficient of $F_n(x)$ (in absolute value). Schur proved that $\lim\sup A_n = \infty$. Emma Lehmer[1] proved that $A_n > cn^{1/3}$ for infinitely many n. In fact she proved that infinitely many such n's are of the form pqr with p, q, and r prime. In the present note we are going to prove that $A_n > n^k$ for every k and infinitely many n. This is implied by the still sharper theorem:

THEOREM 1.[2] *For infinitely many n*

$$A_n > \exp\left[c_1(\log n)^{4/3}\right].$$

Specifically we may take $n = 2\cdot 3\cdot 5 \cdots p_k$ *for sufficiently large* k.

Since

$$\max_{|x|=1} |F_n(x)| \leqq A_n[\phi(n) + 1],$$

Theorem 1 follows at once from the following theorem.

THEOREM 2. *For infinitely many n*

$$\max_{|x|=1} |F_n(x)| > \exp\left[c_2(\log n)^{4/3}\right].$$

For the proof of Theorem 2 we require several lemmas.

LEMMA 1. *Let $f(x)$ be a polynomial of highest coefficient 1 of degree m with all its roots on the unit circle. Suppose that in the unit circle $f(x)$ assumes its maximum at x_0 ($|x_0| = 1$), and let y_0 be the root of $f(x)$ closest to x_0. Then the arc between x_0 and y_0 is not less than π/m; and if it equals π/m, $f(x) = x^m - 1$.*

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[1] Bull. Amer. Math. Soc. vol. 42 (1936) p. 389. Reference to the older literature can be found in this paper.

[2] Throughout the paper c_i denotes a positive constant.