A NOTE ON THE DU BOIS-REYMOND EQUATIONS IN THE CALCULUS OF VARIATIONS

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1. Introduction. McShane [2][1] and Tonelli [4] have given conditions on the integrand function which insure that every absolutely continuous solution of the problem of minimizing the integral

$$I[y] = \int_{x_1}^{x_2} f(x, y, y')dx \tag{1.1}$$

satisfies the du Bois-Reymond form of the Euler equations. The principal purpose of the present note is to give an alternate proof of these results of McShane and Tonelli. In addition to being simpler in detail than the previous proofs, the differentiability theorems of §§2 and 3 involve weaker hypotheses than the corresponding theorems of McShane and Tonelli, both in regard to conditions on the integrand function and in regard to the class of arcs considered. Basically, the present proof is intimately related to the proof of the fundamental lemma as given by Bliss [1, pp. 20–21].

2. A general differentiability theorem. Suppose that for $(x, y, r) = (x, y_1, \cdots, y_n, r_1, \cdots, r_n)$ in a region R consisting of all values (x, y, r) satisfying $x_1 \leqq x \leqq x_2$, y in an open region Δ of $(y_1, \cdots, y_n)$-space, and $r = (r_1, \cdots, r_n)$ arbitrary, the integrand function $f(x, y, r)$ satisfies the following conditions:

(H_1) *For fixed values of* (y, r), $f(x, y, r)$ *is finite and measurable on* x_1x_2;

(H_2) *For fixed values of* x, $f(x, y, r)$ *is of class* C' *in* (y, r).

We shall be interested in proving a differentiability property of an arc $y_i = y_i(x)$ which minimizes $I[y]$ in a class of arcs K. Unquestionably, the case in which K consists of absolutely continuous arcs is the most important. Our argument, however, is not complicated by allowing K to be any prescribed class of arcs which possesses the following properties:

(P_0) *If the arc* $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ *belongs to* K, *then for each* x *on* x_1x_2 *the point* $y = [y_i(x)]$ *is in the region* Δ;

(P_1) *If* $y_i = y_i(x)$ $(x_1 \leqq x \leqq x_2)$ *belongs to* K, *then the functions* $y_i(x)$ *are continuous on* x_1x_2 *and the derivatives* $y_i'(x)$ *exist and are finite a. e.*

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.