DERIVATIVES AND FRÉCHET DIFFERENTIALS

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1. **Generalities.** A function $f(x)$, defined on an open set S of a complex Banach space X, with values in a complex Banach space Y, is said to have a Fréchet differential at a point x_0 of S if for $x=x_0$ the following conditions (G), (D), and (P) are satisfied:

(G) The limit $\lim_{\zeta\to 0}[f(x+\zeta h)-f(x)]/\zeta=\delta_x^h f=\delta f(x, h)$ exists for all h in X; (D) this limit is a continuous linear function of h; (P) the Gâteaux differential $\delta f(x, h)$ is a principal part of the increment, that is, $[f(x+h)-f(x)]-\delta f(x, h)=o(\|h\|)$.

We say that $f(x)$ is F-differentiable on S if these conditions hold at every point of S; if the condition (G) is satisfied for every point in S we call the function G-differentiable on S.

The reader will find in [2][1] or [6] a proof to the effect that a function which is G-differentiable on S—or indeed on more general sets—leads to a function $\delta f(x, h)$ which is linear, in the algebraic sense, with respect to h. We may thus replace the condition (D) by the requirement that the Gâteaux differential be continuous with respect to the argument h, which in turn is equivalent to $\delta f(x, h)$ being $O(1)$, $o(1)$ or $O(\|h\|)$ as $\|h\|$ tends to zero.

Our main purpose is to show that (P) is satisfied automatically if (G) and (D) hold on S, giving a new answer to the question: under which conditions is a G-differentiable function F-differentiable?

Previous solutions of this problem have been of two kinds. The first kind operates with topological conditions on the function $f(x)$, like continuity (see [4]), local boundedness (see [2]), or essential continuity (see [6]). The most general characterization theorem of this type seems to be the following: Let $f(x)$ be G-differentiable on the connected open set S, and bounded on a set $V-M$, where V is a nonvoid open subset of S and M is such that the whole space X is not the sum of a countable number of homothetic images $\alpha_n M+a_n$ of M; under these conditions the function $f(x)$ is F-differentiable on S (see [7]).

A solution of the second kind may be abstracted from [2] or [6]: if the higher differentials $\delta^n f(x; h_1, \cdots, h_n)$ are continuous functions of their h-arguments for one value x_0 of x, then $f(x)$ will be F-differentiable on a suitable neighborhood of x_0. The two kinds of characterizations are rather different; the first type refers to the behaviour

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[1] Numbers in brackets refer to the references cited at the end of the paper.