THE PENTAGON IN THE PROJECTIVE PLANE, WITH A COMMENT ON NAPIER'S RULE

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1. **Introduction.** The relations between the 6 values of the cross ratio of 4 points obtained by permuting the points are well known (§2). The additional relations for the cross ratios of any 4 of 5 collinear points are given in §3. 5 points (or lines) in the plane determine cross ratios with the same relations (4.1). The vertices of a pentagon are on a conic, the sides on another, the pentagon is polar to itself for a third conic. The last may be without real points; in this case the relations between the cross ratios turn out to be equivalent to the famous rule of Napier on the 5 parts of a rectangular spherical triangle, the cross ratios becoming the squares of the sines of the 5 parts. Thus the said relations are a very simple projective formulation of Napier's rule.[1] These considerations form the contents of 4.2. The meets of the diagonals of a pentagon form another pentagon, projective to the first. Repeating the procedure, we obtain a bothways infinite sequence of pentagons, converging to the vertices and sides of the common polar triangle of the 3 above-mentioned conics (4.3). In §5 the relation between the generalized cross ratios of 6 lines (or points) in the plane is determined. Finally we remark that the study of the pentagon in the projective plane is connected, in addition to that of the rectangular spherical triangle, to the study of the triangle in the metrical plane, that is, in elementary Euclidean geometry, and may be important because of its applications to the latter subject.

2. **Cross ratios of 4 elements in 1 dimension.** A point $a=(a_0, a_1)$ on a line is given by a binary linear form $ax=a_0x_0+a_1x_1$ vanishing at the point. Denoting by $[a, b]$ the determinant of the coefficients a, b of two linear forms, the cross ratio of the pairs a, b and c, d is $s=[a:b, c:d]=([a, c]:[a, d]):([b, c]:[b, d])$. Hence $[c:d, a:b]=s$, $[a:b, d:c]=1:s$. By the relation of Ptolemy-Euler-Plücker $[a, b][c, d]+[a, c][d, b]+[a, d][b, c]=0$, there is $[a:c, b:d]=1-s$. Hence, $[b:c, d:a]=[a:d, c:b]=t=s:(s-1)$, $1:s+1:t=1$. $1:s$, $1-s$, and t may be called respectively the *reciprocal*, *complementary*, and *opposite* cross ratios to s. The 6 values s, $1:s$, $1-s$, t, $1:t$, $1-t$ lie, if real,

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[1] And of its dual equivalent, cf, for example, R. E. Moritz, *Napier's theorem for quadrantal triangles*, Congrès International des Mathématiciens, Oslo, 1936, II, pp. 170–171.