# A 5 CURVE THEOREM GENERALIZING THE THEOREM OF CARNOT

TH. MOTZKIN

Because of the 2-dimensionality of the plane a relation holds between any 3 quantities determining a point or a line. The relation between the meets of a line with the sides of a fixed triangle is given by the theorem of Menelaos. This is generalized to the meets of an algebraic curve with a triangle by the theorem of Carnot. I give here a further generalization to the meets of an algebraic curve with a triangle *whose sides are arbitrary algebraic curves.* The formulation of the theorem requires the introduction of the new notion of curve cross ratio; it will be verified by a simple algebraic method allowing us to prove a wide class of allied theorems. The theorem itself is already very extensive; of the numerous special cases, which include many known theorems, I can, in this place, only give a few examples, but I give an outline of the main procedures in obtaining, step by step, special cases and related theorems.

The theorem of Carnot states that the cyclic product of the ratios in which the sides of a polygon are divided by an algebraic curve is 1: it may be written $\prod(AP:BP)=1$, if $A$ is a vertex, $B$ the next vertex, and $P$ a meet of the curve and $AB$. Carnot's theorem, which Poncelet in his epoch-making treatise *Propriétés projectives des figures,* a great part of which is based upon it, calls "le principe fondamental," expresses, as I shall show elsewhere, a characteristic property of algebraic curves; it may also be extended to a class of limiting cases.

The usual proofs of Carnot's theorem are by reduction to one of two special cases. The polygon is decomposed either into triangles or into quadrangles with two opposite vertices in the infinite, and these cases (the second of which is a theorem of Newton) are proved by writing the free term of a polynomial as product of its roots.

Menelaos' theorem is usually proved by homothetic triangles (the simplest case of Newton's theorem); it may also be obtained by dualizing the theorem of Ceva. The latter is a trivial corollary of the representation of a point $P$ by barycentric coordinates, with respect to a triangle $ABC$: if $AP$ meets $BC$ at $A'$, and so on, we have $AC':BC'=ACC':BCC'$ (areas) $=ACP:BCP=ACP:-CBP$; and, evidently, the cyclic product of the ratios analogous to the last is $-1$. Menelaos' theorem, and its exact dual equivalent, may likewise be expressed as identities between areas of triangles, that is, between

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