ON A LEMMA OF LITTLEWOOD AND OFFORD

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Recently Littlewood and Offord[1] proved the following lemma: Let $x_1, x_2, \cdots, x_n$ be complex numbers with $|x_i| \geqq 1$. Consider the sums $\sum_{k=1}^{n} \epsilon_k x_k$, where the ϵ_k are ± 1. Then the number of the sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall into a circle of radius r is not greater than

$$cr2^n(\log n)n^{-1/2}.$$

In the present paper we are going to improve this to

$$cr2^n n^{-1/2}.$$

The case $x_i = 1$ shows that the result is best possible as far as the order is concerned.

First we prove the following theorem.

THEOREM 1. *Let $x_1, x_2, \cdots, x_n$ be n real numbers, $|x_i| \geqq 1$. Then the number of sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall in the interior of an arbitrary interval I of length 2 does not exceed $C_{n,m}$ where $m = [n/2]$. ($[x]$ denotes the integral part of x.)*

Remark. Choose $x_i = 1$, n even. Then the interval $(-1, +1)$ contains $C_{n,m}$ sums $\sum_{k=1}^{n} \epsilon_k x_k$, which shows that our theorem is best possible.

We clearly can assume that all the x_i are not less than 1. To every sum $\sum_{k=1}^{n} \epsilon_k x_k$ we associate a subset of the integers from 1 to n as follows: k belongs to the subset if and only if $\epsilon_k = +1$. If two sums $\sum_{k=1}^{n} \epsilon_k x_k$ and $\sum_{k=1}^{n} \epsilon_k' x_k$ are both in I, neither of the corresponding subsets can contain the other, for otherwise their difference would clearly be not less than 2. Now a theorem of Sperner[2] states that in any collection of subsets of n elements such that of every pair of subsets neither contains the other, the number of sets is not greater than $C_{n,m}$, and this completes the proof.

An analogous theorem probably holds if the x_i are complex numbers, or perhaps even vectors in Hilbert space (possibly even in a Banach space). Thus we can formulate the following conjecture.

CONJECTURE. *Let $x_1, x_2, \cdots, x_n$ be n vectors in Hilbert space, $\|x_i\| \geqq 1$. Then the number of sums $\sum_{k=1}^{n} \epsilon_k x_k$ which fall in the interior of an arbitrary sphere of radius 1 does not exceed $C_{n,m}$.*

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[1] Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp. 277–285.

[2] Math. Zeit. vol. 27 (1928) pp. 544–548.