HERMITIAN QUADRATIC FORMS IN A QUASI-FIELD

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1. Introduction. E. Witt[1] proved the following theorem concerning quadratic forms in a fairly general field:

THEOREM 1. *Let* $f_1 = ax_1^2 + \phi_1(x_2, \cdots, x_n)$ *and* $f_2 = ax_1^2 + \phi_2(x_2, \cdots, x_n)$ *be quadratic forms whose coefficients lie in a given field* F *in which* $2 \neq 0$. *Then the equivalence in* F *of* f_1 *and* f_2 *implies that of* ϕ_1 *and* ϕ_2.

It is our purpose here to generalize this theorem to any quasi-field (a field, except that multiplication may not be commutative) on which is defined a conjugate operation of period 2 with the usual properties

$$\overline{a + b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b} \cdot \bar{a}.$$

Well known examples are any field with $\bar{a} = a$; the field of complex numbers with the usual complex conjugate; the system of quaternions with real coefficients and the usual conjugate. The analogue in a quasi-field of quadratic form in a field is the hermitian quadratic form

$$f = \bar{x}'Ax = \sum_{i,j=2}^{n} \bar{x}_i a_{ij} x_j, \quad \text{where} \quad \overline{A}' = A, \quad \text{or} \quad \bar{a}_{ij} = a_{ji}.$$

The scalars of a quasi-field are the elements s such that $\bar{s} = s$. The diagonal elements of a hermitian matrix are therefore scalars. The process of completing squares is carried out in much the same way as in a field. Thus if, in f above, $a_{11} \neq 0$,

$$f = \left(\bar{x}_1 + \sum_{i=1}^{n} \bar{x}_i a_{i1} a_{11}^{-1}\right) a_{11} \left(x_1 + \sum_{i=2}^{n} a_{11}^{-1} a_{1i} x_i\right)$$
$$+ \sum_{j,k=2}^{n} \bar{x}_j (a_{jk} - a_{j1} a_{11}^{-1} a_{1k}) x_k.$$

Hence the analogue of a form like f_1 in Witt's theorem can be written

$$\bar{x}_1 a x_1 + \phi, \quad \text{where} \quad \phi = \sum_{i,j=2}^{n} \bar{x}_i b_{ij} x_j, \qquad \bar{b}_{ij} = b_{ji}.$$

Since determinants do not exist in a quasi-field (except for hermitian matrices), we cannot demonstrate that a matrix T is nonsingular

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[1] See bibliography.